260 Power Series Solutions Worksheet (solutions) 4/12/20

1. $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \cdots$

2. $(1+x^2)y'' + 6xy' + 6y = 0, \quad y(0) = -1, \quad y'(0) = 1.$

$$0 = (1+x^2)\left[2a_2 + 6a_3 x + 12a_4 x^2 + 20 a_5 x^3 + \cdots\right]$$
$$+ 6x\left[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots\right]$$
$$+ 6\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right]$$

$$= 2a_2 + 6a_0 + x(6a_3 + 6a_1 + 6a_1)$$
$$+ x^2(12a_4 + 2a_2 + 12a_2 + 6a_2) + \cdots$$

$a_0 = y(0) = -1$ and $a_1 = y'(0) = 1$

$0 = 2a_2 + 6a_0 \Rightarrow a_2 = -3a_0 \Rightarrow a_2 = 3$

$0 = 6a_3 + 12a_1 \Rightarrow a_3 = -2a_1 \Rightarrow a_3 = -2$

$0 = 12a_4 + 20a_2 \Rightarrow a_4 = -\frac{5}{3}a_2 \Rightarrow a_4 = -5$

$$y = -1 + x + 3x^2 - 2x^3 - 5x^4 + \cdots$$

3. $(1+8x^2)y'' + 2y' = 0, \quad y(0)=2 \quad y'(0)=-1$

$$0 = (1+8x^2)\left[2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots\right]$$

$$+ 2\left[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots\right]$$

$$= 2a_2 + 2a_1 + x(6a_3 + 4a_2) + x^2(12a_4 + 16a_2 + 6a_3) + \cdots$$

$a_0 = y(0) = 2$ and $a_1 = y'(0) = -1$

$0 = 2a_2 + 2a_1 \Rightarrow a_2 = -a_1 \Rightarrow a_2 = 1$

$0 = 6a_3 + 4a_2 \Rightarrow a_3 = -\frac{2}{3}a_2 \Rightarrow a_3 = -\frac{2}{3}$

$0 = 12a_4 + 16a_2 + 6a_3 \Rightarrow a_4 = -\frac{1}{6}(8a_2 + 3a_3)$

$\Rightarrow a_4 = -\frac{1}{6}(8-2) = -1$

$$y = 2 - x + x^2 - \frac{2}{3}x^3 - x^4 + \cdots$$