Solutions to the worksheet on solving IVPs using Laplace transforms

1 a) $y'' + y' - 2y = 2e^{3t} \qquad y(0) = -1 \quad y'(0) = 4$

$$s^2 L(y) - sy(0) - y'(0) + sL(y) - y(0) - 2L(y) = \frac{2}{s-3}$$

$$s^2 L(y) + s - 4 + sL(y) + 1 - 2L(y) = \frac{2}{s-3}$$

Let $Y = L(y)$

$$(s^2 + s - 2)Y + s - 3 = \frac{2}{s-3}$$

$$Y = \frac{2 - (s-3)^2}{(s-3)(s^2+s-2)} = \frac{2 - (s-3)^2}{(s-3)(s+2)(s-1)}$$

$$= \frac{A}{s-3} + \frac{B}{s+2} + \frac{C}{s-1}$$

Heaviside's method:

$*(s-3),\ s=3. \qquad A = \dfrac{2-0}{5(2)} = \dfrac{1}{5}$

$*(s+2)\ s=-2. \qquad B = \dfrac{2-(-5)^2}{(-5)(-3)} = \dfrac{-23}{15} = -\dfrac{23}{15}$

$*(s-1).\ s=1. \qquad C = \dfrac{2-(-2)^2}{(-2)(3)} = \dfrac{-2}{-6} = \dfrac{1}{3}$

$$y = L^{-1}\left(\frac{1/5}{s-3} - \frac{23/15}{s+2} + \frac{1/3}{s-1}\right)$$

$$= \frac{1}{5}e^{3t} - \frac{23}{15}e^{-2t} + \frac{1}{3}e^{t}$$

1b) $y'' - 4y' + 4y = 1 \quad y(0) = 0 \quad y'(0) = 1$

$$s^2 L(y) - sy(0) - y'(0) - 4[sL(y) - y(0)] + 4L(y) = \frac{1}{s}$$

$$s^2 L(y) - 1 - 4sL(y) + 4L(y) = \frac{1}{s}$$

Let $Y = L(y)$

$$(s^2 - 4s + 4)Y - 1 = \frac{1}{s} \Rightarrow Y = \frac{1+s}{s(s-2)^2}$$

$$\frac{1+s}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$= \frac{A(s-2)^2 + Bs(s-2) + Cs}{s(s-2)^2}$$

The form for this partial fractions decomposition is different from past examples. Heaviside's method doesn't work.

thus $s + 1 = As^2 - 4As + 4A + Bs^2 - 2Bs + Cs$

$$= (A+B)s^2 + (-4A - 2B + C)s + 4A$$

Equate coefficients:

$0 : A + B$

$1 = -4A - 2B + C$

$1 = 4A \Rightarrow A = \frac{1}{4}$

$\Rightarrow B = -\frac{1}{4}$

$\Rightarrow C = 1 + 4A + 2B = 1 + 1 - \frac{1}{2} = \frac{3}{2}$

Solution $y = L^{-1}\left(\frac{1/4}{s} + \frac{-1/4}{s-2} + \frac{3/2}{(s-2)^2}\right) = \frac{1}{4} - \frac{1}{4}e^{2t} + \frac{3}{2}te^{2t}$