Solutions to the worksheet on Laplace and the piecewise continuous forcing function (due 3/27).

1. a) $f(t) = \begin{cases} 1 & \text{if } t < 2 \\ t & \text{if } t \geq 2 \end{cases} = 1 + u(t-2)[t-1]$

$L(f) = \frac{1}{s} + e^{-2s} L(t+2-1) = \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$

b) $f(t) = \begin{cases} e^t & \text{if } t < \ln 2 \\ e^{-t} & \text{if } t > \ln 2 \end{cases} = e^t + u(t - \ln 2)\left[e^{-t} - e^t\right]$

$L(f) = \frac{1}{s-1} + e^{-(\ln 2)s} L\left[e^{-(t+\ln 2)} - e^{t+\ln 2}\right]$

$= \frac{1}{s-1} + e^{-(\ln 2)s} L\left[\frac{1}{2}e^{-t} - 2e^t\right]$

$= \frac{1}{s-1} + e^{-(\ln 2)s}\left(\frac{1/2}{s+1} - \frac{2}{s-1}\right)$ You can stop here, or carry on to the next expression

$= \frac{1}{s-1} + 2^{-s}\left(\frac{1/2}{s+1} - \frac{2}{s-1}\right)$

2. a) $L^{-1}\left(\frac{e^{-s}}{s-2}\right) = u(t-1)e^{2(t-1)} = \begin{cases} 0 & \text{if } t < 1 \\ e^{2(t-1)} & \text{if } t \geq 1 \end{cases}$

From 2nd shifting $e^{-s} L(e^{2t}) = L\left(u(t-1)e^{2(t-1)}\right)$

b) $L^{-1}\left(\frac{1}{s} + \frac{e^{-3s}}{s^2}\right) = 1 + u(t-3)[t-3] = \begin{cases} 1 & \text{if } t < 3 \\ t-2 & \text{if } t \geq 3 \end{cases}$

3. $y'' - 2y' = \begin{cases} 4 & \text{if } t<1 \\ 6 & \text{if } t\geq 1 \end{cases}$ $\quad y(0) = -6,\ y'(0) = 1$

$$= 4 + u(t-1)[6-4] = 4 + u(t-1)[2]$$

a) $s^2 L(y) - sy(0) - y'(0) - 2[sL(y) - y(0)]$
$$= \frac{4}{s} + e^{-s}\left(\frac{2}{s}\right)$$

b) $s^2 Y + 6s - 1 - 2sY - 12 = \frac{4}{s} + e^{-s}\left(\frac{2}{s}\right)$

$(s^2 - 2s)Y + 6s - 13 = \frac{4}{s} + e^{-s}\left(\frac{2}{s}\right)$

$Y = \frac{4}{s^2(s-2)} - \frac{6}{s-2} + \frac{13}{s(s-2)} + e^{-s}\left(\frac{2}{s^2(s-2)}\right)$

$= -\frac{1}{s} - \frac{2}{s^2} + \frac{1}{s-2} - \frac{6}{s-2} + \frac{13/2}{s-2} - \frac{13/2}{s} + e^{-s}\left(\frac{-1/2}{s} - \frac{1}{s^2} + \frac{1/2}{s-2}\right)$

$$y = L^{-1}(Y) = -\frac{15}{2} - 2t + \frac{3}{2}e^{2t} + u(t-1)\left[-\frac{1}{2} - (t-1) + \frac{1}{2}e^{2(t-1)}\right]$$

partial fractions

$\frac{1}{s(s-2)} = \frac{1/2}{s-2} - \frac{1/2}{s}$

$\frac{2}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} = \frac{As(s-2) + B(s-2) + Cs^2}{s^2(s-2)}$

$2 = As^2 - 2As + Bs - 2B + Cs^2 = (A+C)s^2 + (B-2A)s - 2B$

$-2B = 2 \Rightarrow B = -1$ & $B - 2A = 0 \Rightarrow A = -\frac{1}{2}$ & $A + C = 0 \Rightarrow C = \frac{1}{2}$