Worksheet on convolutions and impulses (4/3/20)

1. a) $\int_0^t \tau^7 (t-\tau)^5 \, d\tau = (f * g)(t)$ for $f(t) = t^7$ and $g(t) = t^5$

b) $L(f*g) = L(f)\,L(g) = \frac{7!}{s^8} \cdot \frac{5!}{s^6} = \frac{7!\,5!}{s^{14}}$

c) $h(t) = L^{-1}\left(\frac{7!\,5!}{s^{14}}\right) = L^{-1}\left(\frac{7!\,5!}{13!} \cdot \frac{13!}{s^{14}}\right) = \frac{7!\,5!}{13!} t^{13}$

d) $\int_0^2 \tau^7 (t-2)^5 \, d\tau = h(t) = \frac{7!\,5!}{13!}(2^{13}) \approx 0.7956$

2. $y'' - 4y = 2e^{-t} + 5\delta(t-1)$, $y(0) = -1$, $y'(0) = 2$.

step 1. $y'' - 4y = 2e^{-t}$, $y(0) = -1$, $y'(0) = 2$. Use Laplace transforms just for practice.

$s^2 L(y) - s y(0) - y'(0) - 4 L(y) = \frac{2}{s+1}$

$s^2 Y + s - 2 - 4Y = \frac{2}{s+1}$

$Y = \frac{2}{(s+1)(s^2-4)} + \frac{2-s}{s^2+4} = \frac{-2/3}{s+1} + \frac{1/6}{s-2} + \frac{1/2}{s+2} - \frac{1}{s+2}$

$= \frac{-2/3}{s+1} + \frac{1/6}{s-2} - \frac{1/2}{s+2}$

$\hat{y} = -\frac{2}{3} e^{-t} + \frac{1}{6} e^{2t} - \frac{1}{2} e^{-2t}$

step 2. $w(t) = L^{-1}\left(\frac{1}{s^2-4}\right) = L^{-1}\left(\frac{1/4}{s-2} + \frac{-1/4}{s+2}\right) = \frac{1}{4} e^{2t} - \frac{1}{4} e^{-2t}$

step 3. $y(t) = -\frac{2}{3} e^{-t} + \frac{1}{6} e^{2t} - \frac{1}{2} e^{-2t} + 5\,u(t-1)\left(\frac{1}{4} e^{2(t-1)} - \frac{1}{4} e^{-2(t-1)}\right)$