Solutions to the worksheet on solving linear systems

1. Eigenvalues: $\det\begin{bmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix} = (2-\lambda)(-1-\lambda)-4$

$\lambda = 3, -2$

$$= \lambda^2 - \lambda - 6$$
$$= (\lambda-3)(\lambda+2) = 0$$

Eigenvectors:

$\lambda = 3$: $\begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -x_1 - 4x_2 = 0$

$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ $\quad x_1 = 4 \quad x_2 = -1$

$\lambda = -2$: $\begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -x_1 + x_2 = 0$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad x_1 = 1 \quad x_2 = 1$

General solution: $\vec{y} = c_1\begin{bmatrix} 4 \\ -1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2t}$

Initial values: $\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \vec{y}(0) = c_1\begin{bmatrix} 4 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$4c_1 + c_2 = 2 \Rightarrow c_2 = 2 - 4c_1$

$-c_1 + c_2 = -3$

$\Rightarrow -c_1 + (2-4c_1) = -3$

$\Rightarrow -5c_1 = -5$

$\Rightarrow c_1 = 1$

$\Rightarrow c_2 = 2-4 = -2$

Solution $\vec{y} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}e^{3t} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}e^{-2t}$

2. Eigenvalues: $\det\begin{bmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{bmatrix} = (3-\lambda)(7-\lambda) - (-1)(4)$

$\lambda = 5$

$$= \lambda^2 - 10\lambda + 25$$

$$= (\lambda - 5)^2 = 0$$

Eigenvector: $\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$-x_1 + 2x_2 = 0$ let $x_2 = 1$; $x_1 = 2$

Check: $\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6+4 \\ -2+7 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Other vector: $\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$-u_1 + 2u_2 = 1$. let $u_2 = 0$: $u_1 = -1$

General solution:

$$y = c_1 e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\left(te^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{5t}\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

4. Saddle in #1.
source in #2 (half-spiral)
spiral source in #3.

3. $\vec{y}' = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \vec{y}$

Eigenvalues: $\det \begin{bmatrix} 1-\lambda & 2 \\ -4 & 5-\lambda \end{bmatrix} = (1-\lambda)(5-\lambda) + 8$

$$= \lambda^2 - 6\lambda + 13 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36-52}}{2} = 3 \pm 2i$$

Eigenvectors → only need one

$\lambda = 3+2i$: $\begin{bmatrix} -2-2i & 2 \\ -4 & 2-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2-2i \\ 4 \end{bmatrix}$

$(-2-2i)x_1 + 2x_2 = 0$

Let $x_1 = 2-2i$:

$(-2-2i)(2-2i) + 2x_2 = 0$

$-4-4+2x_2 = 0$

$x_2 = 4$

General solution: $\begin{bmatrix} 2-2i \\ 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{\vec{u}} + i \underbrace{\begin{bmatrix} -2 \\ 0 \end{bmatrix}}_{\vec{v}}$

$\alpha = 3 \quad \beta = 2$

$$\vec{y} = c_1 e^{3t}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} \cos 2t - \begin{bmatrix} -2 \\ 0 \end{bmatrix} \sin 2t\right) + c_2 e^{3t}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} \sin 2t + \begin{bmatrix} -2 \\ 0 \end{bmatrix} \sin 2t\right)$$