SOME (SIMPLE) EXAMPLES OF DIFFERENTIAL EQUATIONS

Example. The SIR model of disease spread looks at the interaction of 3 quantities at time t: $S(t)$ is the number of susceptible individuals, $I(t)$ is the number of infected individuals, and $R(t)$ is the number of recovered (or removed) individuals. The model starts with assumptions about how the disease works:

a) The population remains constant at p_0
b) A constant proportion k of infected individuals recover in each unit of time
c) Each susceptible individual has probability b of a contact that could transmit the disease (if the other person is infected)
d) The number of susceptible individuals decreases as they are infected
e) The number of infected increases as susceptibles are infected and decreases as the infected recover

These assumptions translate into mathematics:

$$\frac{dR}{dt} = kI$$
$$\frac{dS}{dt} = -b\left(\frac{I}{p_0}\right)S$$
$$\frac{dI}{dt} = b\left(\frac{I}{p_0}\right)S - kI$$

The full model is a system of differential equations (and it's a bit too complex for us), so we'll just look at a simple model for $I(t)$ near time $t = 0$ when $S \approx p_0$. This gives a simplified picture that should be reasonably accurate for initial spread of the disease.

When $t \approx 0$, $S \approx p_0$ so $\frac{dI}{dt} = b\left(\frac{I}{p_0}\right)S - kI$

$$\approx b\left(\frac{I}{\cancel{p_0}}\right)\cancel{p_0} - kI$$
$$= I\underbrace{(b-k)}_{\text{constant}}$$

$I' = (b-k)I$

General solution: $I(t) = I_0 e^{(b-k)t}$ where $I_0 = I(0)$ is the initial number of infected individuals.

Improved care increases k.

Social distancing, masks, etc. decrease b

Goal is to have $b < k$.

Date: January 20, 2021.

1. Population tend to grow in proportion to their size: $P'(t) = rP(t)$ where r is a constant growth rate parameter. This means $P(t) = ?$

a) Human population is currently about 7.8 billion and the estimated growth rate is about 1% per year. This means that $P(0) = 7.8$ billion and $P(1) = 1.01p_0$. Use these two facts to solve for r and any other unknown constants.

b) Predict the population in 10 years and in 100 years.

c) What does the model predict in the long-term? Is this realistic?

d) $r = b - d$ where b is a birth rate parameter and d is a death rate parameter. What has to be true of b and d if the population isn't growing exponentially?

$P(t) = Ce^{rt}$

a) $7.8 = P(0) = Ce^0 = C$. Conclude $C = 7.8$

$1.01(7.8) = P(1) = 7.8e^r$

$\Rightarrow 1.01 = e^r$

$\Rightarrow r = \ln(1.01)$. Conclude $P(t) = 7.8e^{\ln(1.01)t}$

$= 7.8(1.01)^t$

b) $P(10) = 7.8(1.01)^{10} \approx 8.6$ billion

$P(100) = 7.8(1.01)^{100} \approx 21.1$ billion

solve $100 = P(t) = 7.8(1.01)^t$ to find that pop. is predicted to reach 100 billion in about 256.4 years.

c) $\lim_{t\to\infty} P(t) = \infty$, which seems problematic (as long as humans only live on earth).

d) If $b > d$, then $\lim_{t\to\infty} P(t) = \infty$, if $b < d$, then $\lim_{t\to\infty} P(t) = 0$.

only if $b = d$ do things remain stable in the long term.

Challenge. The exponential model for population growth $P' = rP$ predicts unbounded population sizes. The logistic model resolves this problem: $P' = aP(1 - bP)$ where a and b are positive constants. If it helps you to solve the problem, you may use $a = 1$ and $b = 1/2$.

a) Solve the integral equation for $P(t)$ (you may need to use the method of partial fractions):

$$\int \frac{1}{P(1-bP)}\, dP = \int a\, dt$$

b) Calculate $\lim_{t\to\infty} P(t)$. Does the logistic model predict unbounded growth?

Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its temperature and the temperature of its environment. As a differential equation: $T' = -k(T - T_m)$ where k is a positive constant of proportionality and T_m is the (constant) temperature of the environment.

2. Solve the differential equation for T.

$T - T_m = Ce^{-kt}$ so $T = T_m + Ce^{-kt}$

3 (CSI Gonzaga). A cooling cup of coffee is found outside on a 5°C day. At 12:15 its temperature is 35°C and at 12:45 its temperature is 25°C.

a) Use the two points to find a formula for $T(t)$, the temperature of the coffee t hours after 12:15 (so $t = 0$ is 12:15).

b) Coffee is usually brewed at about 95°C. Use this to estimate how long ago the coffee was brewed.

$T_m = 5$. measure t in hours.

a) $35 = T(0) = 5 + Ce^{0} \Rightarrow 30 = C$

$25 = T(\frac{1}{2}) = 5 + 30e^{-k(\frac{1}{2})}$

$\Rightarrow \frac{20}{30} = e^{-k/2}$

$\Rightarrow \ln(2/3) = -\frac{k}{2}$

$\Rightarrow k = -2\ln(2/3)$

Conclude: $T = 5 + 30e^{2\ln(2/3)t}$

$= 5 + 30\left(\frac{4}{9}\right)^t$

b) $95 = T(t)$ solve for t.

$95 = 5 + 30e^{2\ln(2/3)t}$

$\Rightarrow 3 = e^{2\ln(2/3)t}$

$\Rightarrow \ln(3) = 2\ln(2/3)t$

$\Rightarrow t = \frac{\ln(3)}{2\ln(2/3)} \approx -1.3548$ hrs

$\Rightarrow$ Conclude: the coffee started cooling about 1 hr. and 21 min. ago. see who bought coffee around 10:54.