LINEAR DIFFERENTIAL EQUATIONS

1 (Completion). We think that the general solution to the differential equation $y' + \left(\frac{3}{x}\right) y = x$ is $y = \frac{x^2}{5} + \frac{c}{x^3}$.

a) Check that this solution actually works (by differentiating and substituting)
b) Solve the IVP: $y' + \left(\frac{3}{x}\right) y = x$, $y(1) = 2$

a) $y = \frac{x^2}{5} + \frac{c}{x^3}$ so $y' = \frac{2x}{5} - \frac{3c}{x^4}$.

$$y' + \left(\frac{3}{x}\right) y = \frac{2x}{5} - \frac{3c}{x^4} + \frac{3}{x}\left(\frac{x^2}{5} + \frac{c}{x^3}\right)$$
$$= \frac{2x}{5} - \frac{3c}{x^4} + \frac{3x}{5} + \frac{3c}{x^4}$$
$$= \frac{5x}{5}$$
$$= x \checkmark$$

b) $2 = y(1) = \frac{1}{5} + \frac{c}{1} \Rightarrow c = 2 - \frac{1}{5} = \frac{9}{5}$

$$y = \frac{x^2}{5} + \frac{9}{5x^3}$$

Theorem. The **general solution** to the homogeneous linear differential equation $y' + p(x)y = 0$ is

$$\boxed{y = ce^{-P(x)}}$$

where $P(x)$ is an antiderivative of $p(x)$ (any function so that $P'(x) = p(x)$; alternatively, $P(x) = \int p(x)\,dx$). Note that the theorem requires that $p(x)$ be integrable.

Theorem. The general solution to the linear differential equation $y' + p(x)y = f(x)$ is

$$\boxed{y = uy_1}$$

where

a) y_1 is any particular solution to the complementary equation $y' + p(x)y = 0$
b) $u = \displaystyle\int \frac{f(x)}{y_1(x)}\,dx$ (add a constant here).

Method. To solve a linear differential equation:

0. Determine that it is linear by rewriting as $y' + p(x)y = f(x)$
1. Solve the complementary equation $y' + p(x)y = 0$ and choose a solution to be y_1
2. Find $u = \displaystyle\int \frac{f(x)}{y_1(x)}\,dx$
3. Your solution is $y = uy_1$ (check by differentiating and substituting)

Date: January 27, 2021.

2 (Graded). Solve the IVP: $y' + 2xy = x$, $y(1) = 1$

Find the general solution to the diff. eq.

$$y' + 2xy = x$$

1. Complementary eqn: $y' + 2xy = 0$

$p(x) = 2x \qquad \int 2x\,dx = x^2 + c$

$P(x) = x^2$

General solution: $y = ce^{-x^2}$

▸ Let $y_1 = e^{-x^2}$

2. $u = \int \frac{x}{e^{-x^2}}\,dx = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$

3. $y = \left(\frac{1}{2}e^{x^2} + c\right)e^{-x^2} = \frac{1}{2} + ce^{-x^2}$

Use the initial condition:

$1 = y(1) = \frac{1}{2} + ce^{-1}$

$\frac{1}{2} = \frac{c}{e}$

$c = \frac{e}{2}$

Solution: $y = \frac{1}{2} + \frac{e}{2}e^{-x^2}$

$$\boxed{y = \frac{1}{2}\left(1 + e^{1-x^2}\right)}$$

3 (Completion). Solve the IVP: $(1+x)y' + 2y = \dfrac{\sin x}{1+x}$, $y(0) = 1$

0. $y' + \dfrac{2}{1+x} y = \dfrac{\sin x}{(1+x)^2}$

$p(x) = \dfrac{2}{1+x}$ $\quad f(x) = \dfrac{\sin x}{(1+x)^2}$

1. Solve comp eq. $y' + \dfrac{2}{1+x} y = 0$

$\int \frac{2}{1+x}\, dx = 2\ln|1+x| + c$

$P(x) = 2\ln|1+x|$

Gen sol'n: $y = c\, e^{-2\ln|1+x|} = c(1+x)^{-2}$

Let $y_1 = (1+x)^{-2}$

2. $u = \int \dfrac{\sin x}{(1+x)^2} \Big/ (1+x)^{-2}\, dx = \int \sin x\, dx$

$= -\cos x + c$

3. $y = (-\cos x + c)(1+x)^{-2}$

Use the initial condition: $1 = y(0) = (-\cos 0 + c)(1+0)^{-2}$

$= -1 + c$

$2 = c$

solution: $y = \dfrac{2 - \cos x}{(1+x)^2}$

Check solution for #2.

$$y = \frac{1}{2}\left(1 + e^{1-x^2}\right)$$

$$y' = -xe^{1-x^2}$$

$$y' + 2xy = -xe^{1-x^2} + 2x\left(\tfrac{1}{2}\right)\left(1 + e^{1-x^2}\right)$$

$$= -xe^{1-x^2} + x + xe^{1-x^2}$$

$$= x \checkmark$$

Check solution for #3.

$$y = \frac{2 - \cos x}{(1+x)^2}$$

$$y' = \frac{\sin x\,(1+x)^2 - (2-\cos x)\,2(1+x)}{(1+x)^4}$$

$$(1+x)y' + 2y = \frac{\sin x\,(1+x)^2 - (2-\cos x)\,2(1+x)}{(1+x)^3} + 2\left[\frac{2-\cos x}{(1+x)^2}\right]$$

$$= \frac{\sin x}{1+x} - \frac{2(2-\cos x)}{(1+x)^2} + \frac{2(2-\cos x)}{(1+x)^2}$$

$$= \frac{\sin x}{1+x} \checkmark$$