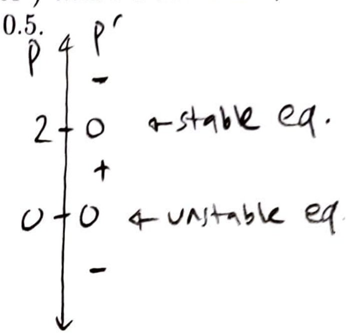
MODELS USING DIFFERENTIAL EQUATIONS

1 (Graded). We saw that the exponential model for population growth $P' = aP$ predicts unbounded population sizes. The logistic model resolves this problem: $P' = aP(1 - bP)$ where a and b are positive constants. If it helps you to solve the problem, you may use $a = 1$ and $b = 0.5$.

a) Draw a phase line and use it to predict the long-term population trends.



Use $P' = P(1-0.5P)$

$0 = P' = P(1-0.5P)$ @ $P=0,\ P=2$

Pop should approach 2.

b) Use separation of variables to find a solution $P(t)$. (I recommend using partial fractions in the integral; you'll also need to use laws of logarithms to solve for P).

$$\frac{1}{P(1-0.5P)} P' = 1 \quad \text{and} \quad \frac{1}{P(1-0.5P)} = \frac{1}{P} + \frac{0.5}{1-0.5P}$$

(partial fractions)

$$\int \frac{1}{P} + \frac{0.5}{1-0.5P}\, dP = \int 1\, dt$$

$$\ln|P| - \ln|1-0.5P| = t + C$$

$$\ln\left|\frac{P}{1-0.5P}\right| = t + C$$

$$\frac{P}{1-0.5P} = Ce^t$$

$$P = Ce^t - \frac{C}{2}Pe^t$$

$$P + \frac{C}{2}Pe^t = Ce^t$$

$$P\left(1 + \frac{C}{2}e^t\right) = Ce^t$$

$$\boxed{P = \frac{Ce^t}{1 + \frac{C}{2}e^t} = \frac{2}{\left(\frac{2}{Ce^t}\right) + 1}}$$

c) Calculate $\lim_{t\to\infty} P(t)$ and compare with your answer for part a.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{2}{\left(\frac{2}{Ce^t}\right) + 1} = 2 \checkmark$$

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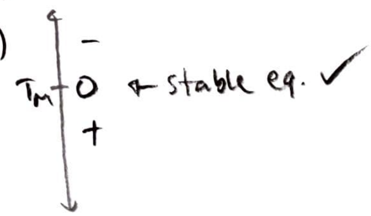
2 (Completion). Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its temperature and the temperature of its environment. As a differential equation: $T' = -k(T - T_m)$ where k is a positive constant of proportionality and T_m is the (constant) temperature of the environment.

a) What is the eventual temperature T of the object? Use your understanding of cooling (or heating) to answer (not math).

b) Draw a phase line for the model and verify that it agrees with your answer for part a.

c) Find the general solution to the differential equation.

a) I think the object will eventually end up at T_m

b)


c) $T' + kT = kT_m$ (linear form)

or $\frac{1}{T - T_m} T' = -k$ (seperated)

Solution: $T = T_m + ce^{-kt}$

3. $212 = T(0) = T_m + C$

$152 = T(1) = T_m + Ce^{-k}$

$112 = T(2) = T_m + Ce^{-2k}$

$60 = C - Ce^{-k} = C(1 - e^{-k})$ (1)

$40 = Ce^{-k} - Ce^{-2k}$
$= e^{-k}[C(1 - e^{-k})]$

3 (Completion). A cup of boiling water (212° F) is placed outside. One minute later the temperature of the water is 152° F. After another minute the temperature is 112° F. What is the outside temperature?

Continued from above;

$40 = e^{-k}(60)$ thus $e^{-k} = \frac{2}{3}$

sub back in to (1) to get $60 = C(1 - \frac{2}{3})$.

Thus $C = 100$.

Now $212 = T_m + C = T_m + 100$.

therefore $T_m = 32°$

4 (Graded). An object with a mass of $m = 10$ kg is launched upward with initial velocity of $v_0 = 60$ m/s. The atmosphere resists the object's motion with a force of 5 Ns/m (that's 5 Newtons for each m/s of speed). The only other force acting on the object is gravity (the acceleration of which is $g = 9.8$ m/s^2 downward). This means that the total force on the object is $F = -mg - 5v$. Newton tells us that $F = ma = mv'$. We now have a differential equation (in the variable v): $\boxed{mv' = -mg - 5v}$

a) Find the terminal velocity of the object (the equilibrium solution to the equation).

b) Find a formula for the velocity of the object.

a) $v' = -g - \frac{5}{m}v = 0$

$-\frac{5}{m}v = g$

$v = -\frac{mg}{5} = -\frac{10}{5}9.8 = -19.6\ \frac{m}{s}$

$\frac{kg(m/s^2)}{(Ns/m)} = \frac{m}{s}$ ✓

b) $v' + \frac{1}{2}v = -9.8$ (linear)

Comp eq: $v' + \frac{1}{2}v = 0$ has solution $v_1 = e^{-\frac{1}{2}t}$

$u = \int \frac{-9.8}{e^{-t/2}}\,dt = \int -9.8e^{t/2}\,dt = -19.6e^{t/2} + C$

General solution: $v = uv_1 = (-19.6e^{t/2} + c)e^{-t/2}$

$v = -19.6 + Ce^{-t/2}$

Use initial value: $60 = v(0) = -19.6 + C$

$C = 79.6$

$v(t) = -19.6 + 79.6e^{-t/2}$