HOMOGENEOUS, LINEAR, SECOND-ORDER EQUATIONS

Definition. The **characteristic polynomial** of the differential equation $ay'' + by' + cy = 0$ is $ar^2 + br + c$ and the **characteristic equation** is $ar^2 + br + c = 0$.

Theorem. If the characteristic polynomial of $ay'' + by' + cy = 0$ has...
a) ...distinct real roots r_1 and r_2, then a general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
b) ...a single (repeated) real root r, then a general solution is $y = e^{rt}(c_1 + c_2 t)$
c) ...complex roots $\lambda \pm i\omega$ (where $\omega > 0$), then a general solution is $y = e^{\lambda t}(c_1 \cos \omega t + c_2 \sin \omega t)$

1 (Graded)**.** Solve the IVP: $y'' - 2y' + y = 0$, $y(0) = 2$, $y'(0) = -1$.

2 (Graded)**.** Solve the IVP: $6y'' - y' - y = 0$, $y(0) = -1$, $y'(0) = 3$.

3 (Completion)**.** An object stretches a spring 0.5 feet at equilibrium. You then compress the spring 4 inches and release it with an initial velocity of 1.5 ft/sec. There's no damping. Your goal is to find a formula for the motion of the spring.
a) Use the equilibrium information ($-mg + \Delta l k = 0$) to solve for k/m. Note that you won't be able to solve for k or m individually (and you don't need to).
b) Use your value for k/m to find a general solution to the differential equation $my'' + ky = 0$.
c) Use the two initial conditions (for $y(0)$ and $y'(0)$) to find the values of the two constants.

The general solution to the undamped spring system $my'' + ky = 0$ is $y = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$. Most of the world prefers to express undamped oscillation as $\boxed{y = R\cos(\omega_0 t - \phi)}$ where

- $\omega_0 = \sqrt{\frac{k}{m}}$
- $R = \sqrt{c_1^2 + c_2^2}$
- ϕ is an angle such that $c_1 = R\cos(\phi)$ and $c_2 = R\sin(\phi)$

4 (Completion)**.** Express your solution to the last problem in the form $y = R\cos(\omega_0 t - \phi)$ (that is, find the values of R, ω_0, and ϕ for your solution).

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