LAPLACE AND THE PIECEWISE CONTINUOUS FORCING FUNCTION

Definition. The **unit step function** is $u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$

Method. The piecewise function $f(t) = \begin{cases} g(t) & \text{if } t < t_1 \\ h(t) & \text{if } t \geq t_1 \end{cases}$ can be expressed as $f(t) = g(t) + u(t - t_1)\left[h(t) - g(t)\right].$

Theorem. If $t_1 \geq 0$ and $L\left(g(t + t_1)\right)$ exists for $s > s_0$, then $\boxed{L\left(u(t - t_1)g(t)\right) = e^{-st_1} L\left(g(t + t_1)\right)}$ for $s > s_0$.

1 (Completion)**.** Find the Laplace transforms of the following functions by first re-expressing them using the unit step function the applying the theorem above.

a) $f(t) = \begin{cases} 1 & \text{if } t < 2 \\ t & \text{if } t \geq 2 \end{cases}$

b) $f(t) = \begin{cases} e^t & \text{if } t < \ln(2) \\ e^{-t} & \text{if } t \geq \ln(2) \end{cases}$

Theorem (Second shifting theorem)**.** $\boxed{L^{-1}\left[e^{-st_1} L\left(g(t)\right)\right] = u(t - t_1)g(t - t_1)}$

2 (Completion)**.** Use the second shifting theorem to find the inverse Laplace transforms of the following functions.

a) $H(s) = \dfrac{e^{-s}}{s - 2}$. Hint: $\frac{1}{s-2} = L(e^{2t})$.

b) $H(s) = \dfrac{1}{s} + \dfrac{e^{-3s}}{s^2}$

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3 (Graded). Use Laplace transforms to solve the IVP: $y'' - 2y' = \begin{cases} 4, & 0 \le t < 1 \\ 6, & t \ge 1 \end{cases}$, $y(0) = -6$, $y'(0) = 1$.