

SOLVING LINEAR SYSTEMS

Method. Let A be an 2×2 matrix. To solve the IVP $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{b}$:

- (1) Find the eigenvalues of A .
- (2) Find an eigenvector for each eigenvalue.
- (3) Use the theorem below to find a general solution.
- (4) Use the initial condition to solve the system of equations for c_1 and c_2 .

Method. The eigenvalues of A are the solutions to $\det(A - \lambda I) = 0$ (solve for λ , often using the quadratic formula). Find the eigenvector(s) corresponding to each eigenvalue λ by solving $(A - \lambda I)\mathbf{x} = 0$ (sub in the eigenvalue and solve for the two entries in \mathbf{x}). In 2-dimensional systems, the solution is usually a line (the eigenspace); to find an eigenvector, just choose a non-zero value for x_1 or x_2 and solve for the other.

Theorem. Let A be an 2×2 matrix. To find a general solution to $\mathbf{y}' = A\mathbf{y}$:

- i) If A has distinct real eigenvalues λ_1, λ_2 with associated **linearly independent** eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, then a general solution is $\mathbf{y}(t) = c_1\mathbf{x}_1e^{\lambda_1 t} + c_2\mathbf{x}_2e^{\lambda_2 t}$.
- ii) If A has an eigenvalue λ with multiplicity of 2 or more and with an associated eigenspace of dimension 1 and \mathbf{x} is any eigenvector, then there are infinitely many vectors \mathbf{u} such that $(A - \lambda I)\mathbf{u} = \mathbf{x}$. If \mathbf{u} is any such vector, then a general solution is $\mathbf{y}(t) = c_1\mathbf{x}e^{\lambda t} + c_2(\mathbf{x}te^{\lambda t} + \mathbf{u}e^{\lambda t})$.
- iii) If A has a complex eigenvalue $\lambda = \alpha + i\beta$ (with $\beta \neq 0$) with associated eigenvector $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, then both \mathbf{u} and \mathbf{v} are nonzero and a general solution is $\mathbf{y}(t) = c_1e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) + c_2e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t)$.

1 (Graded). Solve the IVP $\mathbf{y}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

2 (Completion). Find the general solution to $\mathbf{y}' = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} \mathbf{y}$.

3 (Completion). Find the general solution to $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \mathbf{y}$.

4 (Completion). Plot each system above in Geogebra (link) and characterize each as a source, a sink, a saddle, a spiral source, a spiral sink, an ellipse, or none of the above.