BIFURCATION EVENTS

Sometimes a small change in a parameter in a differential equation can have a big effect on the behavior of the system. This is called a **bifurcation event** (and is a key feature of chaotic systems). We will study the bifurcations that occur in the system

$$\boxed{\mathbf{y}' = \begin{bmatrix} \alpha & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}}$$

1. Find the eigenvalues of $\begin{bmatrix} \alpha & 1 \\ -4 & 0 \end{bmatrix}$ as a function of α (use the quadratic formula).

a) For what values of α does the matrix have distinct real eigenvalues?
b) For what values of α does the matrix have repeated real eigenvalues?
c) For what values of α does the matrix have imaginary eigenvalues?

2. Choose any value of α you want and solve the IVP (note that solutions are hard to find for some choices of α):

$$\mathbf{y}' = \begin{bmatrix} \alpha & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}, \ \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Plot the solution in Desmos (link) and sketch it. You can plot the space curve $\langle x(t), y(t) \rangle = \langle f(t), g(t) \rangle$ by entering `(f(t), g(t))` and making sure the parameter t ranges over a large enough interval to show the distinctive features of the curve. Show any real-valued eigenvectors with a dashed line.

3. Use the GeoGebra phase plane plotter (link) to plot the phase plane for the following values of α. Identify each system as a source, a sink, a saddle, a spiral source, a spiral sink, an ellipse, or none of the above.

a) Your value from the previous problem.
b) $\alpha = -5$.
c) $\alpha = 0$.
d) $\alpha = 2$.
e) $\alpha = 4$.
f) $\alpha = 8$.

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