MATH 301 EXAM 2 NOVEMBER 7, 2014

INSTRUCTIONS: Answer **all of questions 1–5** and **any two of questions 6–9**. Write your solutions on the provided sheets of blank paper. Double check to make sure your name is on the front and everything is clearly labeled, then staple everything together and turn it in. Show any relevant work for problems whose solution is not a proof; solutions without justification may receive little or no credit. As always, write proofs clearly using English words, grammar, and punctuation. You may refer to the following definitions and propositions (as well as any others that you remember).

Definition 1. An integer a is *even* if there is an integer b such that $a = 2b$. Integer a is *odd* if there is an integer b such that $a = 2b + 1$.

Definition 2. Let $a, b \in \mathbb{Z}$. We say that a *divides* b (written $a|b$) if there is an integer c such that $b = ac$.

Definition 3. A natural number n is *prime* if its only positive divisors are 1 and n.

Definition 4. Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $a \equiv b \pmod{n}$ if $n|(a - b)$.

Proposition 1. *Let $a, b \in \mathbb{Z}$ and $p \in \mathbb{N}$. If p is prime and $p|ab$, then $p|a$ or $p|b$.*

Proposition 2. *Let a be an integer. If $2|a$ and $5|a$, then $10|a$.*

Proposition 3. *For any integers a and b, it follows that $a^3 + b^3 = (a + b)^3 \pmod{3}$.*

Answer all of questions 1–5.

1. Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. Prove that if $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{n}$.

2. Prove that $\sqrt{5}$ is irrational. Hint: proposition 1 above shows that for any integer a if $5|a^2$, then $5|a$.

3. Prove that there do not exist three odd integers a, b, and c such that $a^3 + b^3 = c^3$.

4. Let $a \in \mathbb{Z}$. Prove that $a^2|a$ if and only if $a \in \{-1, 0, 1\}$.

5. Let A and B be sets. Prove that if $A - B = \emptyset$, then $A \subseteq B$.

Answer any two of questions 6–9.

6. Determine if the following statement is true or false. Prove it if it is true or provide a counterexample if it is false.

Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a^2 \equiv b^2 \pmod{n}$, then $a \equiv b \pmod{n}$.

7. Determine if the following statement is true or false. Prove it if it is true or provide a counterexample if it is false.

Let a be an integer. If $2|a$ and $6|a$, then $12|a$.

8. Determine if the following statement is true or false. Prove it if it is true or provide a disproof if it is false.

There is an integer a such that $a^2 \equiv -1 \pmod{13}$.

9. Determine if the following statement is true or false. Prove it if it is true or provide a counterexample if it is false.

For all integers a and b, if $a < b$, then $\dfrac{1}{b} < \dfrac{1}{a}$.