PROBLEM 2 OF THE WORKSHEET FROM OCTOBER 17

LOGAN AXON

Definition 1. An integer a is *even* if there is an integer b such that $a = 2b$. The integer a is *odd* if there is an integer b such that $a = 2b + 1$.

Proposition 1. *Let $a \in \mathbb{Z}$. Then a is odd if and only if a^3 is odd.*

As with all proofs of "if and only if" statements we actually need to prove two things:

(1) ($\Rightarrow$) If a is odd, then a^3 is odd;
(2) ($\Leftarrow$) If a^3 is odd, then a is odd.

This means coming up with two proofs, each of which can be a direct proof, a contrapositive proof, or a proof by contradiction. In this case we prove first implication directly and the second by proving the contrapositive.

Proof. First we show that if a is odd, then a^3 is odd. Let a be an odd integer. By definition there is an integer b such that $a = 2b + 1$. Hence $a^3 = (2b + 1)^3 = 8b^3 + 12b^2 + 6b + 1 = 2(4b^3 + 6B^2 + 3b) + 1$. The number $4b^3 + 6B^2 + 3b$ is an integer and thus, by definition, a^3 is odd.

Now we show that if a^3 is odd, then a is odd. We prove the contrapositive: if a is even, then a^3 is even. Let a be an even integer. By definition there is an integer b such that $a = 2b$. Hence $a^3 = (2b)^3 = 8b^3 = 2(4b^3)$. The number $4b^3$ is an integer and thus, by definition, a^3 is even. $\square$

In my original proof I did not use the contrapositive for the second half of the proof. I assumed that a^3 was odd and tried to use that to show that a must also be odd. I applied the definition of odd and found that $a = \sqrt[3]{a^3} = \sqrt[3]{2b+1}$ for some integer b. I then tried to find an expression for $\sqrt[3]{2b+1}$ in terms of b, but made mistakes. My original proof is attached.