Proposition 1. *Let $a \in \mathbb{Z}$. Prove that $a^2|a$ if and only if $a \in \{-1, 0, 1\}$.*

Proof. Let $a \in \mathbb{Z}$ and $a^2|a$. By definition, $\exists x \in \mathbb{Z}$ such that $a^2x = a$. While x is a positive integer, then a^2x will always be greater than a for $a > 1$ and for $a < 1$. While x is negative, then a^2x will always be less than a for $a > 1$ and for $a < 1$. When x is zero, then a could only be 0. Thus the only integers a can be are $\{-1, 0, 1\}$. Let $a \in \{-1, 0, 1\}$. Since $1|(-1)$ and $1|1$ and $0|0$, it can follow that $a^2|a$. □

Proof. First I will show that if $a^2 \mid a$, then $a \in \{-1, 0, 1\}$. Let $a^2 \mid a$. By the definition of divisibility, this means that $a = a^2(b)$ for some integer b. By subtracting a from both sides, we get $0 = a^2b - a$, so $0 = a(ab - 1)$. This means that $a = 0$ or $ab - 1 = 0$. From the equation $ab - 1 = 0$, I get $ab = 1$, which means that a and b are both 1 or a and b are both -1. Thus, $a = 0, a = 1$, or $a = -1$, so $a \in \{-1, 0, 1\}$. Conversely, I must show that if $a \in \{-1, 0, 1\}$, then $a^2 \mid a$. Let $a \in \{-1, 0, 1\}$. Then, for $a = -1$, squaring a gives $a^2 = 1$. For $a = 1$, squaring both sides gives $a^2 = 1$. Finally, for $a = 0$, squaring both sides gives $a^2 = 0$, so it follows that $a^2 \mid a$ for $a = -1, a = 1$, and $a = 0$. Thus, if $a \in \{-1, 0, 1\}$, then $a^2 \mid a$. □

Proof. First we must show that if $a^2|a$ then $a \in \{-1, 0, 1\}$. Suppose $a \in \mathbb{Z}$ and $a^2|a$. By definition, there exists an $x \in \mathbb{Z}$ such that $a^2x = a$. Hence, $ax = 1$ or $a = 0$. Thus, $a \in \{-1, 0, 1\}$ because either $a|1$ or $a = 0$.

Now we must show that if $a \in \{-1, 0, 1\}$, then $a^2|a$. Suppose $a \in \mathbb{Z}$ and $a \in \{-1, 0, 1\}$.

Case 1: $a = -1$. Then $a^2 = 1$. Hence, $a = -1 = (-1)(1) = a^2(-1)$. Thus, $a^2|a$.

Case 2: $a = 0$. Then $a^2 = 0$. Since any number multiplied by zero is equal to zero, a is divisible by any number, including a^2

Case 3: $a = 1$, Then $a^2 = 1$. Hence, $a = 1 = (1)(1) = a^2(1)$. Thus, $a^2|a$. Therefore, $a^2|a$ if and only if $a \in \{-1, 0, 1\}$. □

Proposition 2. *Let A and B be sets. If $A - B = \emptyset$, then $A \subseteq B$.*

Proof. Suppose A and B are sets and $A - B = \emptyset$. Then $A \cap \neg B = \emptyset$. This means that there are no elements in A that are not in B. Hence, every element contained within A is contained within B. Therefore, by definition, $A \subseteq B$. □

Proof. Suppose A and B are sets and $A - B \neq \emptyset$. Then there must be some $a \in A - B$. By definition $a \in A$ and $a \notin B$. Therefore $A \nsubseteq B$. □

Proposition 3. *Let $a, b, \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a^2 \equiv b^2 \pmod{n}$, then $a \equiv b \pmod{n}$.*

Proof. This statement is false. Disproof. Let n=6, a=4, and b=2. Then, $4^2 \equiv 2^2 \pmod 6$ is true because $6|\ (a^2 - b^2)$ since $6|\ 12$. But, it is not the case that $4 \equiv 2 \pmod 6$ because $6 \nmid (4 - 2)$ by the definition of divisibility. Therefore, the proposition is false. □

Proposition 4. *$\sqrt{5}$ is irrational.*

Proof. Suppose $\sqrt{5}$ is rational. Then there exists 2 integers x and y such that $\sqrt{5} = \frac{x}{y}$. Suppose that x and y do not have any common denominators. Then $5 = \frac{x^2}{y^2}$, $y^2 = 5x^2$ and $x^2 = 5y^2$. Thus both x and y are divisible by 5. This is a contradiction because we stated that there are no common denominators between x and y. Thus $\sqrt{5}$ is irrational. □

Proof. Let $\sqrt{5}$ be rational. By definition then, $\sqrt{5} = \frac{a}{b}$ for some integers a and b of which the fraction $\frac{a}{b}$ is irreducible. Thus $5 = \frac{a^2}{b^2}$ which leads to $5b^2 = a^2$. By Proposition 1, this means $5 \mid a$. Due to this, $5x = a$ for some integer x. It follows logically then that $5b^2 = (5x)^2$. Hence $b^2 = 5x^2$. Again by Proposition 1, this means $5 \mid b$. However, this means that both $5 \mid a$ and $5 \mid b$. This would mean that a and b share the common factor 5, which contradicts the assumption that $\frac{a}{b}$ is irreducible. Thus by contradiction, $\sqrt{5}$ is irrational. □

Proposition 5. *There does not exist three odd integers a, b, and c such that $a^3 + b^3 = c^3$.*

The following is a proof by contrapositive.

Proof. Suppose there exist three odd integers a, b, and c such that $a^3 + b^3 \neq c^3$. Let $a = 2l + 1$, $b = 2m + 1$ and $c = 2n + 1$. It follows that $(2l + 1)^3 + (2m + 1)^3 = (2n + 1)^3$. This equality is equivalent to the following equality $2(4l^3 + 6l^2 + 3l + 4m^3 + 6m^2 + 3m) = 2(4n^3 + 6n^2 + 3n) + 1$. Let $y = (4l^3 + 6l^2 + 3l + 4m^3 + 6m^2 + 3m)$ and $x = (4n^3 + 6n^2 + 3n)$. The quantities x and y are integers such that $2x = 2y + 1$. By definition an odd number does not equal an even number. Thus we have shown that there exist three odd integers a, b, and c such that $a^3 + b^3 \neq c^3$. Therefore by contrapositive we have shown that there do not exist three odd integers a, b, and c such that $a^3 + b^3 = c^3$ □

Proof. Suppose $a, b \in \mathbb{Z}$ such that a and b are both odd. By definition of odd numbers, there is $x, y \in \mathbb{Z}$ such that $a = 2x + 1$ and $b = 2y + 1$. By plugging these into the equation you get $(a^3) + (b^3) = (2x + 1)^3 + (2y + 1)^3 = (8x^3 + 12x^2 + 6x + 1) + (8y^3 + 12y^2 + 6y + 1) = 8x^3 + 8y^3 + 12x^2 + 12y^2 + 6x + 6y + 2$ $= 2(4x^3 + 4y^3 + 6x^2 + 6y^2 + 3x + 3y + 1)$. By definition of an even number, it shows that $a^3 + b^3$ is an even number. Thus c must be even if both a and b are odd.Therefore there do not exist three odd numbers such that $a^3 + b^3 = c^3$. □

Proof. Let a, b, and c be odd integers. As such, $a = 2x + 1, b = 2y + 1$, and $c = 2z + 1$ for some integers x, y, and z. Thus $a^3 = 8x^3 + 12x^2 + 6x + 1, b^3 = 8y^3 + 12y^2 + 6y + 1$, and $c^3 = 8z^3 + 12z^2 + 6z + 1$. Hence $a^3 + b^3 = (8x^3 + 12x^2 + 6x + 1) + (8y^3 + 12y^2 + 6y + 1)$. This number is even as $a^3 + b^3 = 2[4(x^3 + y^3) + 6(x^2 + y^2) + 3(x + y) + 1]$. In contrast, c^3 is odd as $c^3 = 2(4z^3 + 6z^2 + 3z) + 1$. As $a^3 + b^3$ and c^3 are different parities, they can not be equal. Therefore $a^3 + b^3$ does not equal c^3. □

Proposition 6. *Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. Prove that if $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{n}$.*

Proof. Let $a, b \in \mathbb{Z}, n \in \mathbb{N}$, and $a \equiv b \pmod{n}$. By Definition 2 and Definition 1, $nx = a - b$ for some integer x. In addition, $a^2 - b^2 = (a + b)(a - b)$. As $nx = a - b$, it follows that $a^2 - b^2 = (a + b)nx$. Thus $a^2 - b^2 = ny$ for some integer y where $y = (a + b)x$. Therefore $a^2 \equiv b^2 \pmod{n}$. □

Proposition 7. *All dogs are the same color.*

Proof. We will use induction to prove that for every n, all the dogs in a set of n dogs are the same color.

Base case: $n = 1$. In this case there is just one dog in the set, so all dogs in the set are the same color.

Now suppose that all dogs in a set of size n must be the same color. Consider a set of $n + 1$ dogs. Choose any dog in this set and remove it from the set. We are left with a set of n dogs and thus the remaining n dogs are all the same color. Add the first dog back to the set and remove any other dog. Again we are left with a set of n dogs and therefore all of them are the same color. This includes the first dog we removed and therefore all $n = 1$ dogs are the same color. □