Name(s):

Math 301 | Counting and Proving | September 29, 2014

Definition 1. For any $n \in \mathbb{N}$ the factorial of n is $n! = n(n-1)(n-2)\ldots(2)(1)$.

1. Calculate $2!$, $3!$, $4!$, and $5!$

2. How many ways are there to arrange the given numbers (without repetition)?

a) $1, 2$

b) $1, 2, 3$

c) $1, 2, 3, 4$

d) $1, 2, 3, 4, 5$

3. How many ways are there to arrange the numbers $1, 2, 3, \ldots, n$ (without repetition)?

Definition 2. Let $n, m \in \mathbb{Z}$.

- n is *even* if there is $a \in \mathbb{Z}$ such that $n = 2a$.
- n is *odd* if there is $a \in \mathbb{Z}$ such that $n = 2a + 1$.
- m *divides* n (written $m|n$) if there is $a \in \mathbb{Z}$ such that $ma = n$.
- If $n \geq 2$ and the only divisors n are 1 and n, then n is *prime*.
- If $n \geq 2$ and n is not prime, then n is *composite*.

4. Let $x, y \in \mathbb{Z}$.

a) Prove that if either x or y is even, then xy is even.

b) State the converse of "if either x or y is even, then xy is even". Note that the converse has the same implicit quantifier(s) as the original statement.

c) Is the converse true or false?

5. Let $x, y, z \in \mathbb{Z}$.

a) Prove that if $z|x$ or $z|y$, then $z|(xy)$.

b) State the converse of "if $z|x$ or $z|y$, then $z|(xy)$".

c) Either prove the converse or find an example showing that it is not true.

6. Let $a, b, c \in \mathbb{Z}$. Prove that if $a|b$ and $a|c$, then $a|(b+c)$.

7. Let $a, b, c \in \mathbb{Z}$. Prove that if $a|b$ and $a|(b+c)$, then $a|c$.

8. Prove that if $n \in \mathbb{N}$ and $n \geq 2$, then $n!+2$, $n!+3$, $n!+4$, and $\ldots$, and $n!+n$ are all composite. (This means that $n!+2$, $n!+3$, $n!+4$, $\ldots$, $n!+n$ is a sequence of $n-1$ consecutive composite numbers, thus showing that there are arbitrarily large gaps between prime numbers).

9. Prove that if two integers have the same parity (both odd or both even), then their sum is even. Hint: use cases.

10. Prove that every odd integer is the difference of two squares (e.g. $5 = 3^2 - 2^2$).