1. Make a truth table for the expression $P \implies (P \vee Q)$.

Solution. This statement is a tautology. Your truth table should show this.

2. Write the following expression (Bertrand's postulate) in English:

$$(\forall n \in \mathbb{N})\,(\exists p \in \mathbb{N})\ \left[(p \text{ is prime}) \wedge (n < p < 2n)\right].$$

Solution. For any natural number n there is a prime between n and $2n$.

3. Is it true that $\left\{x \in \mathbb{R} : x^2 \leq 1\right\} \subseteq \left\{x^2 : x \in \mathbb{R}\right\}$?

Solution. No; $\left\{x \in \mathbb{R} : x^2 \leq 1\right\} = [-1, 1]$ and $\left\{x^2 : x \in \mathbb{R}\right\} = [0, \infty)$.

4. Find all integer solutions to the equation $x^2 \equiv 1 \pmod 3$.

Solution. Every integer that is not a multiple of 3 is a solution. The set of solutions is $\{a \in \mathbb{Z} : 3 \nmid a\}$.

5. Prove that if 1 is even, then 0.5 is an integer.

Solution. Suppose 1 is even. By definition there is an integer a such that $2a = 1$. Hence $a = \frac{1}{2}$. Therefore $\frac{1}{2}$ is an integer.

6. Let $a, b \in \mathbb{Z}$. Prove that if ab is even, then either a or b is even.

Solution. Proof of the contrapositive: if a and b are both odd, then ab is odd. Let $a, b \in \mathbb{Z}$ both be odd. By definition there are integers x and y such that $a = 2x + 1$ and $b = 2y + 1$. Hence $ab = (2x + 1)(2y + 1) = 2(2xy + x + y) + 1$. The quantity $2xy + x + y$ is an integer and therefore ab is odd.

7. Let $a \in \mathbb{N}$. Prove that if $2^a - 1$ is prime, then a is odd or $a = 2$.

Solution. Proof of the contrapositive: if a is even and $a \neq 2$, then $2^a - 1$ is not prime. Let $a \in \mathbb{N}$ be even with $a \neq 2$. By definition there is an integer b such that $a = 2b$. Because $a \neq 2$, we know that $a \geq 4$ and thus $b \geq 2$. Hence $2^b - 1$ and $2^b + 1$ are both integers greater than 1. This is relevant because $2^a - 1 = 2^{2b} - 1 = (2^b - 1)(2^b + 1)$. Therefore $2^a - 1$ is not prime.

8. Is the converse of the statement in problem 7 true?

Solution. The converse: if a is odd or $a = 2$, then $2^a - 1$ is prime. False: $7 | (2^9 - 1)$.
For more on this see http://primes.utm.edu/mersenne/.

9. Determine if the following statement is true or false:

$$(\forall a, b \in \mathbb{N})\,(\exists d \in \mathbb{N})\,\left[(d|a \wedge d|b) \wedge ((\forall x \in \mathbb{N})\,(x|a \wedge x|b) \implies x \leq d)\right].$$

Solution. The statement claims that a and b have a greatest common divisor. This is true.