PORTFOLIO PROOFS

The first section contains definitions relevant to problems in the following sections. Section 2 contains statements for you to prove (or, in subsection D, either prove or disprove).

1. Definitions

Definition 1. $\lim_{x\to a} f(x) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x) - L| < \epsilon.$$

Definition 2. Let $a, b \in \mathbb{N}$. The **least common multiple** of a and b is the smallest natural number divisible by both a and b (and it is written $\operatorname{lcm}(a,b)$).

Definition 3. Integers a and b are **relatively prime** if their only common divisors are 1 and -1.

Definition 4. The Fibonacci sequence is defined recursively by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The sequence is thus $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$.

2. Proofs

A. **Direct and contrapositive proofs.**

A.1. Let $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.

A.2. Let $x \in \mathbb{R}$. If $x > 0$, then $x + \dfrac{1}{x} \geq 2$.

A.3. Suppose $n \in \mathbb{Z}$. If n is odd, then $8 \mid (n^2 - 1)$.

A.4. Use the definition of the limit (1) to prove that $\displaystyle\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$.

A.5. Use the definition of the limit (1) to prove that $\displaystyle\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \neq 0$.

A.6. Prove that if $n \in \mathbb{N}$ and $n \geq 2$, then the numbers $n! + 2$, $n! + 3$, $n! + 4$, $\ldots$, $n! + n$ are all composite. (This means that $n! + 2$, $n! + 3$, $n! + 4$, $\ldots$, $n! + n$ is a sequence of $n - 1$ consecutive composite numbers, thus showing that there are arbitrarily large gaps between prime numbers).

B. **Proofs by contradiction and non-conditional statements.**

B.1. The cube root of 3 is irrational.

B.2. Suppose $a, b, p \in \mathbb{Z}$ and p is prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

B.3. For every integer n, at least one of n, $n+1$, or $n+2$ is divisible by 3. (This requires a detailed proof, not the informal argument we used in class on October 12).

B.4. For any natural numbers a and b, $a = \operatorname{lcm}(a,b)$ if and only if $b \mid a$.

B.5. Let C be a circle in $\mathbb{R}^2$ centered at $(1,1)$. Then either $(2,3) \notin C$ or $(-2,2) \notin C$.

C. **Proofs and disproofs.** Determine if the statement is true and either prove or disprove it.

C.1. There are integers m and n such that $m^2 + mn + n^2$ is a perfect square.

C.2. If $n \in \mathbb{Z}$, then $4 \nmid (n^2 - 3)$.

C.3. There is a natural number n such that $11 \mid (2^n - 1)$.

C.4. Suppose A, B, and C are sets. If $A \times C \subseteq B \times C$, then $A \subseteq B$.

C.5. For any sets A, B, and C, $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

D. **Induction I.**

D.1. For any $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$.

D.2. Any two successive Fibonacci numbers are relatively prime (see definitions 3 and 4).

D.3. Prove that $(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$ for every $n \in \mathbb{N}$.

E. **Induction II.**

E.1. Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.

E.2. Consider the 2×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that for any $n \in \mathbb{N}$,

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

(where F_k is the k^{th} term of the Fibonacci sequence 4).

E.3. Define a new function on the positive real numbers:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Prove that if $n \in \mathbb{N}$, then $\Gamma(n+1) = n!$ (so that this function is a version of the factorial for non-integers; interestingly, $\Gamma(1/2) = \sqrt{\pi}$). Hint: integration by parts.

F. **Uncategorized proofs.**

F.1. Every odd integer is the difference of two squares.

F.2. Let $a, b \in \mathbb{Z}$ and let $d = \gcd(a, b)$. Then $\{ma + nb : m, n \in \mathbb{Z}\} = \{dn : n \in \mathbb{Z}\}$.

F.3. Let $n \in \mathbb{N}$. Then any set of n integers has a subset whose sum is divisible by n.

F.4. Prove that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(x, y) = 2^{x-1}(2y - 1)$ is a bijection.