MATH 301 REVIEW

A. Fundamentals

A.1. **Sets.** Chapter 1. Set-builder notation, Cartesian product, intersection, union, difference, complement, subsets, power sets.

A.2. **Logic.** Chapter 2. Statements, logical connectives, truth tables, **quantifiers**, **negations**, converse, contrapositive.

B. Advanced fundamentals

B.1. **Relations.** Chapter 11. Reflexivity, symmetry, transitivity, equivalence relations, partitions.

B.2. **Functions.** Chapter 12. Injective, surjective, bijective, composition, inverse.

C. Proof techniques

C.1. **Prove $P \implies Q$.** Chapters 4-6. Remember that there's an implicit universal quantifier.

Direct proof. Suppose P. ... Therefore Q. $\square$

Contrapositive proof. Suppose $\neg Q$. ... Therefore $\neg P$. $\square$

Proof by contradiction. Suppose P and $\neg Q$. ... Arrive at some obvious contradiction. $\square$

C.2. **Prove $P \iff Q$.** Chapter 7. Prove $P \implies Q$ and $Q \implies P$ using any technique of C.1.

C.3. **Prove $\exists x, P(x)$.** Chapter 7. Usually give an example, the more concrete the better.

C.4. **Disproof.** Chapter 9. Prove the negation of the statement. Often a counterexample (a kind of existence proof).

C.5. **Induction.** Chapter 10. Used for statements of the form $\forall n \in A, S_n$ where $A \subseteq \mathbb{N}$ (usually). Examples below have $A = \mathbb{N}$. Formalizes recursive arguments.

Proof by induction. Base case: Prove S_1.
Inductive step: Let $k \in \mathbb{N}$ and suppose S_k. ... Therefore S_{k+1}. $\square$

Proof by strong induction. Base case: Prove S_1 and S_2 and ... and S_n where n is large enough that your inductive argument will work.
Inductive step: Let $k \in N$ with $k \geq n$ and suppose $S_1 \wedge S_2 \wedge \cdots \wedge S_k$. ... Therefore S_{k+1}. $\square$

The inductive step is always a proof of a conditional statement:
a) $S_k \implies S_{k+1}$ or
b) $(S_1 \wedge S_2 \wedge \cdots \wedge S_k) \implies S_{k+1}$.

Use the techniques of C.1 and **state your inductive hypothesis clearly**

D. Special techniques for sets and functions

Chapter 8 for sets and Chapter 12 for functions.

D.1. **Prove $A \subseteq B$.**

Proof. Let $a \in A$. ... Therefore $a \in B$. (Element chasing). $\square$

D.2. **Prove $A = B$.** Prove $A \subseteq B$ and $B \subseteq A$.

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D.3. **Prove $f : A \to B$ is injective.**

Proof. Suppose $x, y \in A$ and $f(x) = f(y)$. ... Therefore $x = y$. □

D.4. **Prove $f : A \to B$ is surjective.**

Proof. Let $b \in B$. Prove that there is $a \in A$ such that $f(a) = b$. (A kind of existence proof). □

D.5. **The pigeonhole principle.** And the connection of functions to the relative sizes of sets generally. See the last two worksheets: *Bijections and cardinalities* and *Functions and the pigeonhole principle (and some geometry).*

E. Sample questions

1. Let $A = \{\sqrt{x} : x \in \mathbb{Q}\}$. Which of the following are elements of the set $\mathcal{P}(A)$? Mark all correct answers.

a) $1/\sqrt{2}$
b) 0
c) $\emptyset$
d) $\mathbb{N}$
e) $\mathbb{Q}$

2. Consider the following statement:

> If f is differentiable and f and f' are both strictly increasing functions on the interval $(0, \infty)$, then $\lim_{x\to\infty} f(x) = \infty$.

The following argument is suggested as a proof of this statement.

(1) By the Mean Value Theorem, there is a number c_1 in the interval $(1, 2)$ such that

$$f'(c_1) = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1) > 0.$$

(2) For each $x > 2$, there is a number c_x in the interval $(2, x)$ such that $f'(c_x) = \dfrac{f(x) - f(2)}{x - 2}$.
(3) The function f' is strictly increasing, hence $f'(c_x) > f'(c_1)$.
(4) For each $x > 2$, it then follows that $f(x) > f(2) + (x - 2)f'(c_1)$.
(5) Therefore $\lim_{x\to\infty} f(x) = \infty$.

Which of the following statements is true? Mark only one answer.

a) The argument is not valid since the hypotheses of the Mean Value Theorem are not satisfied in (1) and (2).
b) The argument is not valid since (3) is not valid.
c) The argument is not valid since (4) cannot be deduced from the previous steps.
d) The argument is not valid since (4) does not imply (5).
e) The argument is valid.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be any function and let $a, L \in \mathbb{R}$. Identify the correct negation of the statement:

> For each number $\epsilon > 0$, there is a number $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.

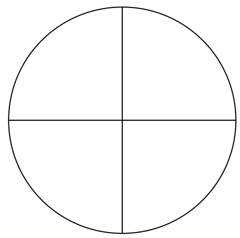
Mark only one answer.

a) For each number $\epsilon > 0$, there is a number $\delta > 0$ such that if $|x - a| \geq \delta$, then $|f(x) - L| \geq \epsilon$.
b) For each number $\epsilon > 0$, there is a number $\delta > 0$ and a number x such that $|x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.
c) There is a number $\epsilon > 0$ such that for each number $\delta > 0$, if $|x - a| \geq \delta$, then $|f(x) - L| \geq \epsilon$.
d) There is a number $\epsilon > 0$ such that for each number $\delta > 0$ there is a number x such that $|x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.
e) There is a number $\epsilon > 0$ and there is a number $\delta > 0$ and there is a number x such that $|x - a| \geq \delta$ and $|f(x) - L| \geq \epsilon$.

4. Let P, Q, and R be statements. Which of the following is logically equivalent to $P \implies (Q \wedge R)$? Mark all correct answers.

a) $(\neg Q \wedge \neg R) \implies \neg P$
b) $(\neg Q \vee \neg R) \implies \neg P$
c) $(Q \wedge R) \implies P$
d) $P \vee \neg(Q \wedge R)$
e) $\neg P \vee (Q \wedge R)$

5. A circular region is divided into 4 sectors as shown below. Twenty-one points are chosen in the circular region, none of which lie on the sector edges. Which of the following statements must be true no matter how the points are chosen? Mark all correct answers.



a) Some sector contains at least 6 points.
b) Some sector contains at least 7 points.
c) Some sector contains at most 4 points.
d) Some pair of adjacent sectors contains a total of at least 11 points.
e) Some pair of adjacent sectors contains a total of at least 12 points.

6. Identify the best translation of the statement into English. Mark only one answer.

$$\boxed{\forall x, y \in \mathbb{R}, (x < y \implies \exists q \in \mathbb{Q}, x < q < y)}$$

a) For all real numbers x and y, there is is a rational number q such that $x < q < y$.
b) For all real numbers x and y, if every rational number q is between x and y, then $x < y$.
c) For all real numbers x and y, if $x < y$, then there is a rational number q.
d) If x and y are distinct real numbers, then there is a rational number between x and y.
e) If x and y are distinct real numbers, then $x < y$.

7. Prove that $\sqrt{6}$ is irrational.

8. Prove that the product of any 4 consecutive integers is divisible by 24.

9. Let $n \in \mathbb{N}$. Prove that if $8 \nmid (n^2 - 1)$, then n is even.

10. Determine if the statement is true or false and provide either a proof or disproof:

For every positive $x \in \mathbb{Q}$, there is $y \in \mathbb{Q}$ such that $0 < y < x$.

11. Determine if the statement is true or false and provide either a proof or disproof:

There are no integers a and b such that $20a + 8b = 14$.