Proof?

Theorem

If a and b are real numbers, then $a + b \geq 2\sqrt{ab}$.

Proof.

Suppose, by way of contradiction, that $a, b \in \mathbb{R}$ and $a + b < 2\sqrt{ab}$. It follows that

$$(a+b)^2 < \left(2\sqrt{ab}\right)^2.$$

Thus

$$a^2 + 2ab + b^2 < 4ab.$$

Consequently

$$(a-b)^2 = a^2 - 2ab + b^2 < 0.$$

This is a contradiction since a real number squared must be at least 0. □

Perfect numbers

Definitions

A natural number n is **perfect** if it is equal to the sum of its proper divisors.

Examples:

$$\begin{aligned} 6 &= 1 + 2 + 3 \\ 28 &= 1 + 2 + 4 + 7 + 14 \\ 496 &= 1 = 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 \end{aligned}$$

Non-examples:

$$\begin{aligned} 4 &\neq 1 + 2 = 3 \\ 9 &\neq 1 + 3 = 4 \\ 12 &\neq 1 + 2 + 3 + 4 + 6 = 16 \end{aligned}$$

The Euclid-Euler Theorem

Definitions

- P is the set of perfect numbers;
- $S = \left\{2^{(n-1)}\left(2^n - 1\right) : n \in \mathbb{N} \text{ and } 2^n - 1 \text{ is prime}\right\}$;
- E is the set of even numbers.

Theorem (Euclid-Euler)

$S = P \cap E$.

$S \subseteq P \cap E$

Theorem

$S \subseteq P \cap E$.

Proof.

Let $a \in S$. Then $a = 2^{(n-1)}(2^n - 1)$ with $2^n - 1$ is prime.

- Why is $a \in E$?
- Why is $a \in P$? (Hint: use the formula $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$ to add the divisors of a).

We have shown that $a \in E$ and $a \in P$. It follows that $a \in P \cap E$ (definition of intersection). Therefore $S \subseteq P \cap E$. □