INDUCTION II ... AND RELATIONS

1. Find the flaw in the following proof.

Theorem. *If $n \in \mathbb{N}$, then n is odd*

Proof by strong induction. Base case: $n = 1$. Then n is clearly odd.

Now suppose that $k \in \mathbb{N}$ and that if $l \in \mathbb{N}$ and $l \leq k$, then l is odd. Consider that $k + 1 = (k - 1) + 2$ and that $k - 1 \leq k$. By IH $k - 1$ is odd. Hence $k - 1 = 2a + 1$ for some integer a. We now have

$$\begin{aligned} k + 1 &= (k - 1) + 2 \\ &= (2a + 1) + 2 \\ &= 2(a + 1) + 1. \end{aligned}$$

Since $a + 1$ is an integer, we have shown that $k + 1$ is odd.

Therefore, by strong induction, every natural number is odd. □

2. Complete any necessary bases cases in the following proof.

Theorem. *If $n \in \mathbb{N}$, then $12 | (n^4 - n^2)$.*

Proof by strong induction. We will defy tradition and begin with the inductive step. Let $k \in \mathbb{N}$ and suppose that if $l \in \mathbb{N}$ and $l \leq k$, then $12 | (l^4 - l^2)$. We wish to prove that $12 | \left((k + 1)^4 - (k + 1)^2\right)$.

Let $l = k - 5$. Then $l \leq k$, so by IH $12 | (l^4 - l^2)$. Hence there is an integer a such that $l^4 - l^2 = 12a$. Then

$$\begin{aligned} (k + 1)^4 - (k + 1)^2 &= (l + 6)^4 - (l + 6)^2 \\ &= l^4 + 24l^3 + 216l^2 + 864l + 1296 - (l^2 + 12l + 63) \\ &= (l^4 - l^2) + 24l^3 + 216l^2 + 852l + 1260 \\ &= 12a + 24l^3 + 216l^2 + 852l + 1260 \\ &= 12(a + 2l^3 + 18l^2 + 71l + 105). \end{aligned}$$

It follows that $12 | \left((k + 1)^4 - (k + 1)^2\right)$.

We now must establish the base cases to complete the proof.

□

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Definition. Let R be a relation on a set A.

- R is **reflexive** if $\forall x \in A,\ xRx$.
- R is **symmetric** if $\forall x, y \in A,\ xRy \implies yRx$.
- R is **transitive** if $\forall x, y, z \in A,\ (xRy \wedge yRz) \implies xRz$.

3. Which of the three properties in the definition does the relation of $<$ (on $\mathbb{R}$) have?

4. Which of the three properties in the definition does the relation of divides (on $\mathbb{Z}$) have?

5. Which of the three properties in the definition does the relation of congruence modulo 5 (on $\mathbb{Z}$) have?