BIJECTIONS AND CARDINALITIES

If there is a bijection between sets A and B, then the sets must be the same size. This turns out to give a good definition of the size of a set.

Definition. Let A and B be sets. Then we say A and B have the same **cardinality** and write $|A| = |B|$ if there is a bijection between A and B.

Example. Let $A = \{a, b, c\}$ and let $B = \{1, 2, 3\}$. Then there is a bijection between A and B, for example $\{(a, 1), (b, 2), (c, 3)\}$. Thus $|A| = |B|$ by the definition above.

The definition works nicely for finite sets. Things get a little stranger when dealing with infinite sets.

1. Let $A = \{2n : n \in \mathbb{Z}\}$ and $B = \{2n + 1 : n \in \mathbb{Z}\}$. Follow the model of the example above to establish that:

a) $|A| = |B|$

b) $|A| = |\mathbb{Z}|$

c) $|\mathbb{N}| = |\mathbb{Z}|$

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Theorem (Shröder-Bernstein)**.** *Let A and B be sets. If there is an injective function from A to B and another injective function from B to A, then there is a bijection between A and B.*

2. We can use the Shröder-Bernstein theorem to prove that $|\mathbb{Z}| = |\mathbb{Q}|$. We just need to:

a) Produce an injective function from $\mathbb{Z}$ to $\mathbb{Q}$.

b) Produce an injective function from $\mathbb{Q}$ to $\mathbb{Z}$.

3. Prove that $|\mathbb{N}| = |\mathbb{Q}|$.

4. Do you think all infinite sets have the same cardinality?