FUNCTIONS AND THE PIGEONHOLE PRINCIPLE (AND SOME GEOMETRY)

1. Let A, B, and C be sets. Suppose $f : A \to B$ and $g : B \to C$ are both surjective. Finish proving that $g \circ f : A \to C$ is surjective.

Proof. Let A, B, and C be sets. Suppose $f : A \to B$ and $g : B \to C$ are both surjective. If $C = \emptyset$, then $g \circ f$ is trivially surjective. To deal with the other case, suppose $C \neq \emptyset$. Let $c \in C$.

Thus $(g \circ f)(a) = c$. Therefore $g \circ f$ is surjective. □

We saw on the last worksheet that functions have connections with the sizes of sets.

Definition. Let A and B be sets.

a) $|A| = |B|$ if there is a bijection between A and B.
b) $|A| \leq |B|$ if there is an injective function from A to B.

Another connection is the **pigeonhole principle**:

Theorem (Pigeonhole principle). *Suppose A and B are sets and $f : A \to B$ is any function.*

a) If $|A| > |B|$, then f is not injective.
b) If $|A| < |B|$, then f is not surjective.

An informal statement of part a of the pigeonhole principle (for finite sets): if $m, n \in \mathbb{N}$ and $m < n$, then if we put n things into m containers, we must have at least one container with two or more things in it. I hope this sounds obvious; it turns out some non-obvious theorems can be proved using this principle.

2. Prove that if each point in $\mathbb{R}^2$ is colored in one of two colors, then there must be two points of the same color exactly one unit apart. Hint: Consider the corners of an equilateral triangle.

Challenge. Let $n \in \mathbb{N}$. Then any set of n integers has a subset whose sum is divisible by n.

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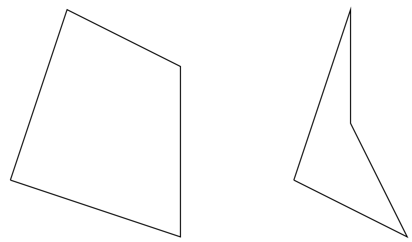
The next problem is an investigation into geometry that doesn't use the pigeonhole principle.

3. A well-known geometry theorem states that this sum of the interior angles of a triangle must be 180°. This problem guides you through finding a similar formula for any convex polygon (with $3, 4, 5, \ldots$ sides). Convex means that every interior angle is less than 180°, so the first quadrilateral below is convex, but the second isn't.[1]



a) Determine the sum of the interior angles of a convex quadrilateral (hint: divide it into two triangles).

b) Determine the sum of the interior angles of a convex pentagon (hint: divide it into a triangle and a quadrilateral, then use your formula from part a.

c) Find a general formula for the sum of interior angles of a convex polygon with $n = 3, 4, 5, \ldots$ sides, then prove that the formula always works.

[1]Technically, convex means that the straight line between any two points of the polygon is entirely inside the polygon