

THE CANTOR SET

Definition. We construct the Cantor middle-thirds set as follows. Define the following sets (as unions of intervals in \mathbb{R}):

$$\begin{aligned}C_0 &= [0, 1] \\C_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\C_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\&\vdots\end{aligned}$$

In general, C_{k+1} is formed by removing the open middle third of each interval in C_k . The **Cantor middle-thirds set** is

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k$$

1. Prove or disprove the following statements about \mathcal{C} .

- a) $\mathcal{C} = \emptyset$
- b) $\forall n \in \mathbb{N}, \left(0, \frac{1}{2^n}\right) \not\subseteq \mathcal{C}$
- c) $\forall n \in \mathbb{N}, \left(0, \frac{1}{2^n}\right) \cap \mathcal{C} = \emptyset$

Definition. Any real number can be expressed in ternary, which is like binary but with 3 possible values for each digit. For example, the ternary number 201 is

$$(2 \times 3^2) + (0 \times 3^1) + (1 \times 3^0) = 18 + 0 + 1 = 19.$$

And the ternary number 0.201 is

$$(2 \times 3^{-1}) + (0 \times 3^{-2}) + (1 \times 3^{-3}) = \frac{2}{3} + \frac{0}{9} + \frac{1}{27} = \frac{19}{27}.$$

2. Prove that $x \in \mathcal{C}$ if and only if x can be written as a ternary number using only the digits 0 and 2 (no 1s allowed).

3. Is $1/4$ in the Cantor set?