Definition. The function f is **bounded above** if there is a number M such that $f(x) \leq M$ for every x in the domain of f. The function f is **bounded below** if there is a number m such that $f(x) \geq m$ for every x in the domain of f. The function f is **bounded** if it is bounded both above and below.

1. Prove that there is a bounded function (on $\mathbb{R}$).

2. Prove that the function $f(x) = 1 - x^2$ is bounded above.

3. Prove that the function $f(x) = 1 - x^2$ is not bounded.

4. The citizens of Königsberg are proud of their bridges. Those with leisure time like to walk across all of the bridges and then end up back where they started. Is it possible to do this and cross each bridge exactly once? Does it matter where they start?

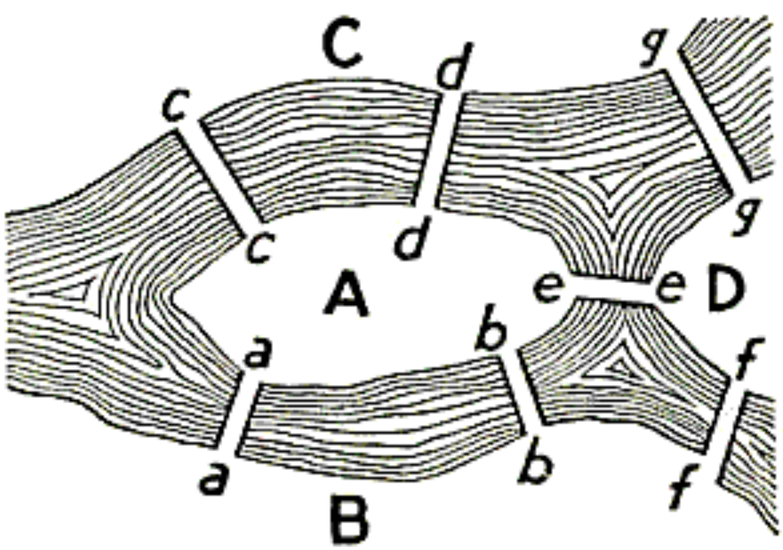

Figure 98. *Geographic Map: The Königsberg Bridges.*

Theorem (Mean Value Theorem)**.** *If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a number c with $a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Definition. A function g is **strictly increasing** on the interval (a, b) if for all numbers c and d, $a < c < d < b$ implies that $g(c) < g(d)$.

5. Consider the following statement:

If f is differentiable on the interval $(0, \infty)$ and f and f' are both strictly increasing on $(0, \infty)$, then $\lim_{x\to\infty} f(x) = \infty$.

The following argument is suggested as a proof of this statement.

(1) By the Mean Value Theorem, there is a number c_1 in the interval $(1, 2)$ such that

$$f'(c_1) = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1) > 0.$$

(2) For each $x > 2$, there is a number c_x in the interval $(2, x)$ such that $f'(c_x) = \dfrac{f(x) - f(2)}{x - 2}$.

(3) The function f' is strictly increasing, hence $f'(c_x) > f'(c_1)$.

(4) For each $x > 2$, it then follows that $f(x) > f(2) + (x - 2)f'(c_1)$.

(5) Therefore $\lim_{x\to\infty} f(x) = \infty$.

Which of the following statements is true? Mark only one answer.

a) The argument is not valid since the hypotheses of the Mean Value Theorem are not satisfied in (1) and (2).

b) The argument is not valid since (3) is not valid.

c) The argument is not valid since (4) cannot be deduced from the previous steps.

d) The argument is not valid since (4) does not imply (5).

e) The argument is valid.