Proposition 1. *Let $x \in \mathbb{R}$. If $x > 0$, then $x + \frac{1}{x} \geq 2$.*

Proof. Proof by contradiction. Assume, by contradiction, that x is a real number where $x > 0$ and $x + (1/x) < 2$. By subtracting 2 from both sides we get $x + (1/x) - 2 < 0$, which can also be written as $(x^2 + 1 - 2x)/x < 0$. This is equal to $(x-1)^2/x < 0$. Multiplying both sides by x we get $(x-1)^2 < 0$. This is a contradiction, as $(x-1)^2 \geq 0$ for every real number $x > 0$. Therefore if $x > 0$, then $x + (1/x) \geq 2$. □

Comments:

Proposition 1. *Let $x \in \mathbb{R}$. If $x > 0$, then $x + \frac{1}{x} \geq 2$.*

Proof. Suppose, by way of contradiction, that $x > 0$ and $x + \frac{1}{x} < 0$.

Then, by placing both expressions over a common denominator $\frac{x^2+1}{x} < 0$.

Hence $x^2 + 1 < 0$.

Since $x > 0$, $x^2 + 1$ cannot be negative, so this statement is a contradiction.

Therefore, $x + \frac{1}{x} \geq 0$. □

Comments:

Proposition 1. *Let $x \in \mathbb{R}$. If $x > 0$, then $x + \dfrac{1}{x} \geq 2$.*

Proof. Let x be a real number and suppose x is greater than 0. Multiplying by x on both sides of the equation, $x + \frac{1}{x}$ is greater than or equal to 2, yields, $x^2 + 1$ is greater than or equal to $2x$. Subtracting $2x$ from one side of the equation then reveals, $x^2 - 2x + 1$ is greater than or equal to 0. Then factoring this equation simplifies it to, $(x - 1)^2$ is greater than or equal to 0. The value of x may be any real number greater than 0, meaning the only equality is found at $x = 1$, and every other possible real number is allowed. Therefore, $x + \frac{1}{x}$ is greater than or equal to 2.

Let x be a real number and suppose x is greater than 0. Multiplying by x on both sides of the equation, $x + \frac{1}{x} \geq 2$, yields, $x^2 + 1 \geq 2x$. Subtracting $2x$ from one side of the equation then reveals, $x^2 - 2x + 1 \geq 0$. Then factoring this equation simplifies it to, $(x - 1)^2 \geq 0$. The value of x may be any real number greater than 0, meaning the only equality is found at $x = 1$, and every other possible real number is allowed. Therefore, $x + \frac{1}{x} \geq 2$. □

Comments:

Proposition 1. *Let $x \in \mathbb{R}$. If $x > 0$, then $x + \dfrac{1}{x} \geq 2$.*

Proof. Let $x \in \mathbb{R}$ and suppose $x > 0$. Consider the quantity $(x - 1)$. We know that $(x - 1)$ is negative when $0 < x < 1$, 0 when $x = 1$, and positive when $x > 1$. It then follows that $(x - 1)^2 \geq 0$. Hence $x^2 - 2x + 1 \geq 0$. And $x^2 + 1 \geq 2x$. Then, you can divide both sides by x without flipping the direction of the inequality since x is strictly positive, resulting in $x + \frac{1}{x} \geq 2$. Therefore, $x + \frac{1}{x} \geq 2$. □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Assume a is odd. Since a is odd $a = 2k + 1$, $k \in \mathbb{Z}$. So $a^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 4(k^2 + k) = 4(k(k+1))$. By definition one of k and $k+1$ is even and one is odd. Thus the product $k(k+1)$ is even. Because $k(k + 1)$ is even it is divisible by two so we can write it as $2k(k + 1)/2$, and it is still an integer. We get that $8|(4 * 2k(k + 1)/2)$. Therefore $8 * (k(k + 1)/2 = (4 * 2k(k + 1)/2)$ so by definition $8|(a^2 - 1)$. □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Let $a \in \mathbb{Z}$. Suppose a is odd. By definition of odd, there exists an integer c such that $a = 2c + 1$. By definition of divides, there exists an integer d such that $8d = (a^2 - 1)$.

$$a^2 - 1 = (2c + 1)^2 - 1 = 4c^2 + 4c + 1 - 1 = 4c^2 + 4c = 4c(c + 1)$$

So, $c(c + 1)$ is an even integer. Using the definition of divides, $a^2 - 1 = 4 * 2 * c(c + 1)/2$. Therefore, $8|(a^2 - 1)$. □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Let $n \in \mathbb{Z}$, and suppose that n is odd. By definition (of odd) there is an integer k such that $n = 2k + 1$. Hence $n^2 - 1 = (2k + 1)^2 - 1 = (4k^2 + 4k + 1) - 1 = 4k^2 + 4k = 4(k^2 + k)$. Consider the following two cases:

Case 1. k is even: Then there is $c \in \mathbb{Z}$ such that $k = 2c$. Then $k^2 = (2c)^2 = 4c^2 = 2(2c^2)$. So $k^2 + k = 2(2c^2) + 2c = 2(2c^2 + c)$, which is even.

Case 2. k is odd: Then there is $t \in \mathbb{Z}$ such that $k = 2t + 1$. Then $k^2 = (2t + 1)^2 = 4t^2 + 4t + 1$. So $k^2 + k = (4t^2 + 4t + 1) + (2t + 1) = 4t^2 + 6t + 2 = 2(2t^2 + 3t + 1)$, which is even.

In either case, $k^2 + k$ is even. By definition (of even) there is an integer s such that $k^2 + k = 2s$. Remember: $n^2 - 1 = 4(k^2 + k)$. By substitution we have $n^2 - 1 = 4(2s) = 8s$. Since $s \in \mathbb{Z}$, then $8|(n^2 - 1)$. Therefore if n is odd, then $8|(n^2 - 1)$. □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Suppose a $\in \mathbb{Z}$ and a is odd. Since a is odd, there exists a c $\in \mathbb{Z}$ such that a = 2c + 1 by definition. Therefore, $a^2 = (2c + 1)^2 = 4c^2$ + 4c + 1 = 4(c^2 + c) + 1. Then, 8 | 4(c^2 + c) + 1 - 1, so then 2 | c^2 + c. Since a is odd and a = 2c + 1, then c must be even. Since c is even, then c^2 must be even. Since c and c^2 are both even, then 2 | c^2 + c by definition. Therefore, 8 | (a^2 - 1) □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Let a be an integer and suppose a is odd. If a is odd, then by definition of odds there is an integer c such that $a = 2c+1$. Substituting this into $8|(a^2-1)$ we get $8|((2c+1)^2-1) = 8|((4c^2+4c+1)-1) = 8|(4c(c+1))$. Where $c(c+1)$ is the product of two consecutive integers which is the product of an odd and an even. And in class we proved the product of an odd and an even to be even. Then by definition of evens, there exists an integer n such that $c(c+1) = 2n$. Substituting that into $8|(4c(c+1))$, we get $8|(4(2n)) = 8|8n$. We know that an integer divides multiples of itself, therefore, 8 divides (a^2-1) when a is odd. □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Let a be an integer, and suppose a is odd. By definition of odd numbers, a equals 2c + 1, where c is an integer. Then, $(a^2-1) = (2c+1)^2 - 1 = 4c^2 + 4c = 2(2c^2+2c)$. Thus, by definition, and where $2c^2 + 2c$ is an integer, a^2 is even. Also by definition there is an integer d such that $8d = a^2 - 1$. Thus, $8|(a^2-1)$ □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Lemma 1. *k^2 has the same parity as k for all $k \in \mathbb{Z}$*

Proof. Let $k \in \mathbb{Z}$
Case 1: k is odd
By the definition of odd, $k = 2n + 1, n \in \mathbb{Z}$ It follows that

$$k^2 = (2n + 1)(2n + 1)$$
$$k^2 = 4n^2 + 4n + 1$$
$$k^2 = 2(2n^2 + 2n) + 1$$

Since $(2n^2 + 2n) \in \mathbb{Z}$, k^2 is odd by the definition of odd.

Case 2: k is even
By the definition of even, $k = 2n, n \in \mathbb{Z}$ It follows that

$$k^2 = (2n)(2n)$$
$$k^2 = 4n^2$$
$$k^2 = 2(2n^2)$$

Since $(2n^2) \in \mathbb{Z}$, k^2 is odd by the definition of even.
By Case 1 and Case 2, $\forall k \in \mathbb{Z}$: k^2 and k have the same parity □

Lemma 2. *$k^2 + k$ is even for all $k \in \mathbb{Z}$*

Proof. Let $k \in \mathbb{Z}$ Case 1: k is even
It follows that k^2 is even by the proof that k^2 and k have the same parity **??**. So by the definition of even, $k = 2n, k^2 = 2mn, m \in \mathbb{Z}$ It follows that

$$k^2 + k = 2m + 2n$$
$$k^2 + k = 2(m + n)$$

Since $(m + n) \in \mathbb{Z}$, $k^2 + k$ is even when k is even
Case 2: k is odd
It follows that k^2 is odd by the proof that k^2 and k have the same parity **??**. So by the definition of even, $k = 2n + 1, k^2 = 2m + 1n, m \in \mathbb{Z}$ It follows that

$$k^2 + k = 2m + 1 + 2n + 1$$
$$k^2 + k = 2m + 2n + 2$$
$$k^2 + k = 2(m + n + 1)$$

Since $(m + n + 1) \in \mathbb{Z}$, $k^2 + k$ is even when k is odd
By Case 1 and Case 2, $k^2 + k$ is odd $\forall k \in \mathbb{Z}$ □

Proof. let $a \in \mathbb{Z}$ Suppose (by way of contradiction) 8 does not divide $a^2 - 1$ and a is odd So, $\nexists c \in \mathbb{Z}$: $8c = a^2 - 1$ by the definition of divides and $a = 2k + 1$ by the definition of odd.
It follows that $a^2 - 1 = (2k + 1)(2k + 1) - 1$ and thus, $a^2 - 1 = 4k^2 + 4k$, so $a^2 - 1 = 4(k^2 - k)$. It follows from previous work in proving Lemma **??**, that $k^2 - k$ is even for all $k \in \mathbb{Z}$, so by the definition of even, $a^2 - 1 = 4(2(j))$ for some $j \in \mathbb{Z}$. So, $a^2 - 1 = 8j$, and thus by the definition of divides, $8|a^2 - 1$, but this contradicts our assumption that 8 does not divide $a^2 - 1$ when a is odd, therefore $8|a^2 - 1$ when a is odd. □

Comments:

(continue on the reverse)

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Let some integer a be odd. We know, by the Division Algorithm, that there exist unique integers q and r such that $a = 4q + r$ where $0 \leq r < 4$. When r is 0, $a = 4q = 2(2q)$ and when r is 2, $a = 4q + 2 = 2(2q + 1)$. Hence, a is even whenever $r = 0$ or $r = 2$. However, when $r = 1$, $a = 4q + 1 = 2(2q) + 1$ and when $r = 3$, $a = 4q + 3 = 2(2q + 1) + 1$. So, a is odd whenever $r = 1$ or $r = 3$. And since we are assuming that a is odd, we can uniquely express a as $a = 4q + r$ with $r = 1$ or $r = 3$.
We will now consider $a^2 - 1$ for the two cases.

Case 1. $r = 1$
When $r = 1$, $a = 4q + 1$. It follows that $a^2 - 1 = (4q + 1)^2 - 1 = 16q^2 + 8q + 1 - 1 = 16q^2 + 8q$. But an 8 can be factored out of that equation, so $a^2 - 1 = 8(2q^2 + q)$. And since $2q^2 + q$ is an integer, by definition $8|(a^2 - 1)$.

Case 2. $r = 3$
When $r = 3$, $a = 4q + 3$. Consequently, $a^2 - 1 = (4q + 3)^2 - 1 = 16q^2 + 24q + 9 - 1 = 16q^2 + 24q + 8$. But an 8 can be factored out of that equation, so $a^2 - 1 = 8(2q^2 + 3q + 1)$. And since $2q^2 + 3q + 1$ is an integer, by definition $8|(a^2 - 1)$.

$8|a^2 - 1$ in both cases. Therefore, if a is odd, then $8 \mid (a^2 - 1)$. □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Suppose $a \in \mathbb{Z}$. Let a be odd. By definition, $a = 2b+1$ for some $b \in \mathbb{Z}$. Then, $a^2-1 = (2b+1)^2-1 = 4b^2 + 4b + 1 - 1 = 4b(b+1)$. Because b could be either odd or even, we need two cases to look at.
Case 1: Let b be odd. By definition, $b = 2c + 1$ for some $c \in \mathbb{Z}$. Then, $b + 1 = 2c + 1 + 1 = 2c + 2$ is even. It follows that $2c + 2 = 2d$ for $d = c + 2$. Therefore, $a^2 - 1 = 4b(b + 1) = 4b(2c + 2) = 4b(2d) = 8bd$. Consequently, 8 divides $a^2 - 1$.
Case 2: Let b be even. By definition, $b = 2n$ for some $n \in \mathbb{Z}$. Then, $a^2 - 1 = 4b(b + 1) = 4(2n)(b + 1) = 8n(b + 1)$. Therefore, 8 divides $a^2 - 1$.
Since 8 divides $a^2 - 1$ in both cases, it is true that 8 divides $a^2 - 1$ for all $a \in \mathbb{Z}$. □

Comments:

Lemma 3. *Let $a, b \in \mathbb{Z}$. If a and b have opposite parity, then ab is even (this was proved in class).*

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Let a be an integer and suppose a is odd.
By definition, there is an integer b such that $a = 2b + 1$
Then, $a^2 - 1 = (2b + 1)^2 - 1 = 4b^2 + 4b + 1 - 1 = 4b^2 + 4b = 4(b^2 + b) = 4(b)(b + 1)$.
Because $b \in \mathbb{Z}$, we know that either b or $b + 1$ is even, and we know that the product of an even and odd integer is even.
Thus, by definition, there is an integer c such that $(b)(b+1) = 2c$, so $a^2 - 1 = 4(2c) = 8c$. We can see that $8|8c$. Therefore, $8|(a^2 - 1)$. □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Let $a \epsilon \mathbb{Z}$, and suppose a is odd. By definition of odd, $\exists m \epsilon \mathbb{Z}$ such that $2m + 1 = a$. It follows that $a^2 - 1 = ((2m + 1) + 1)((2m + 1) - 1) = (2m + 2)(2m)$. This can be written as $4m^2 + 4m$, and factored as $4m(m + 1)$. From here, there are 2 cases we need to consider.
Case 1: Suppose m is odd. By definition of odd, $\exists x \epsilon \mathbb{Z}$ such that $2x + 1 = m$. It follows that $m + 1 = (2x + 1) + 1 = 2x + 2 = 2(x + 1)$, and since $2|(2(x + 1))$, we know that $m + 1$ is even. This means that the quantity $m(m + 1)$ must then also have 2 as a factor, so we know that $2|(m + 1)$.
Case 2: Suppose m is even. By definition of even, $2|m$, so by similar logic, we know that $2|(m(m + 1))$.
In either case, $2|(m(m+1))$. By multiplication of 4, we can see that $8|(4m(m+1))$. Therefore $8|(a^2-1)$. □

Comments:

Proposition 2. *Suppose $a \in \mathbb{Z}$. If a is odd, then $8 \mid (a^2 - 1)$.*

Proof. Suppose that a is an odd integer. By definition of odd, there is an integer n such that $a = 2n + 1$ Consider $a^2 - 1 = (2n + 1)^2 - 1$. This statement can be simplified to $a^2 - 1 = 4n(n + 1)$. Since n and $n + 1$ are two consecutive integers, it follows that one must be odd and one must be even. Due to this, it follows that $4n(n + 1)$ is divisible by 8, as the divisors 4 and 2 must be present. Therefore by definition of odd and divides, if a is odd, then $8 \mid (a^2 - 1)$. □

Comments:

Proposition 2. *Suppose* $a \in \mathbb{Z}$. *If* a *is odd, then* $8 \mid (a^2 - 1)$.

Proof. Let $a \in \mathbb{Z}$ and suppose a is odd. By definition, there is an integer x for which $a = 2x + 1$. Then $a^2 = 4x^2 + 4x + 1$. Thus $a^2 - 1 = 4x^2 + 4x$. So $a^2 - 1 = 4(x)(x+1)$. Since x and $x+1$ have opposite parity, the product of x and $x+1$ is even too*. Then by definition, there is an integer y for which $(x)(x+1) = 2y$. Hence $a^2 - 1 = 4(2y)$. Thus $a^2 - 1 = 8y$. Therefore, $8|(a^2 - 1)$. □

Comments: