Perfect numbers

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Examples:

$$\begin{aligned} 6 &= 1 + 2 + 3 \\ 28 &= 1 + 2 + 4 + 7 + 14 \\ 496 &= 1 = 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 \end{aligned}$$

Non-examples:

$$\begin{aligned} 4 &\neq 1 + 2 = 3 \\ 9 &\neq 1 + 3 = 4 \\ 12 &\neq 1 + 2 + 3 + 4 + 6 = 16 \end{aligned}$$

Who cares?

- Philo of Alexandria ($\sim$ 20 BCE–50 CE), a Jewish philosopher, claimed the world was created in 6 days that the moon orbits in 28 days because 6 and 28 are perfect.
- Greek mathematicians like Nichomachus ($\sim$ 100 CE) knew the perfect numbers on the last slide and the fourth: 8128.
- Egyptian mathematician Ismail ibn Fallūs (1194–1252 CE) knew the next three perfect numbers: $33,550,336$; $8,589,869,056$; and $137,438,691,328$.
- Europeans were interested by the 15$^{\text{th}}$ century when an unknown mathematician (re)discovered $33,550,336$.

And Euclid

Theorem (Euclid)

Let $n \in \mathbb{N}$*. If* $2^n - 1$ *is prime, then* $2^{(n-1)}(2^n - 1)$ *is perfect.*

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Let $n \in \mathbb{N}$. If $2^n - 1$ is prime, then $2^{(n-1)}\left(2^n - 1\right)$ is perfect.

Definition

We define the following sets:

- P is the set of perfect numbers;
- $S = \left\{2^{(n-1)}\left(2^n - 1\right) : n \in \mathbb{N} \text{ and } 2^n - 1 \text{ is prime}\right\}$;

Question

What relationship does Euclid's theorem establish between S and P?

Ibn al-Haytham and Euler too

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- E is the set of even numbers.

Conjecture (Ibn al-Haytham, $\sim$ 1000 CE)

$(P \cap E) \subseteq S$.

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Proved by Euler in the 18$^{\text{th}}$ century.

Theorem (Euclid-Euler)

$S = P \cap E$.

Open questions

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Question

Are there any odd perfect numbers?