INDUCTION II ... AND RELATIONS

1. Find the flaw in the following proof.

Proposition 1. *If $n \in \mathbb{N}$, then n is odd*

Proof by strong induction. Base case: $n = 1$. Clearly n is odd.

Now suppose that $k \in \mathbb{N}$ and that if $l \in \mathbb{N}$ and $l \leq k$, then l is odd. Consider that $k+1 = (k-1)+2$ and that $k-1 \leq k$. By IH $k-1$ is odd. Hence $k-1 = 2a+1$ for some integer a. We now have

$$\begin{aligned} k+1 &= (k-1)+2 \\ &= (2a+1)+2 \\ &= 2(a+1)+1. \end{aligned}$$

Since $a+1$ is an integer, we have shown that $k+1$ is odd.

Therefore, by strong induction, every natural number is odd. □

2. Complete all the base cases needed to make the following proof work.

Theorem. *If $n \in \mathbb{N}$, then $12|(n^4 - n^2)$.*

Proof by strong induction. We will defy tradition and begin with the inductive step. Let $k \in \mathbb{N}$ and suppose that if $l \in \mathbb{N}$ and $l \leq k$, then $12|(l^4 - l^2)$. We wish to prove that $12|\left((k+1)^4 - (k+1)^2\right)$.

Let $l = k-5$. Then $l \leq k$, so by the inductive hypothesis $12|(l^4 - l^2)$. Hence there is an integer a such that $l^4 - l^2 = 12a$. Then

$$\begin{aligned} (k+1)^4 - (k+1)^2 &= (l+6)^4 - (l+6)^2 \\ &= l^4 + 24l^3 + 216l^2 + 864l + 1296 - (l^2 + 12l + 63) \\ &= (l^4 - l^2) + 24l^3 + 216l^2 + 852l + 1260 \\ &= 12a + 24l^3 + 216l^2 + 852l + 1260 \\ &= 12(a + 2l^3 + 18l^2 + 71l + 105). \end{aligned}$$

It follows that $12|\left((k+1)^4 - (k+1)^2\right)$.

We now must establish the base cases to complete the proof.

□

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Definition. Let R be a relation on the set A.

a) R is **reflexive** if $\forall x \in A,\ xRx$.
b) R is **symmetric** if $\forall x, y \in A,\ xRy \implies yRx$.
c) R is **transitive** if $\forall x, y, z \in A,\ (xRy \land yRz) \implies xRz$.

3. Which of the three properties in the definition does the relation of $<$ (on $\mathbb{R}$) have?

4. Which of the three properties in the definition does the relation of divides (on $\mathbb{Z}$) have?

5. Which of the three properties in the definition does the relation of congruence modulo 5 (on $\mathbb{Z}$) have?

Challenge. Prove that you are correct in problems 3-5.