FUNCTIONS

1. Let A, B, and C be sets. Suppose $f : A \to B$ and $g : B \to C$ are both surjective. Finish proving that $g \circ f : A \to C$ is surjective.

Proof. Let A, B, and C be sets. Suppose $f : A \to B$ and $g : B \to C$ are both surjective. If $C = \emptyset$, then $g \circ f$ is trivially surjective. To deal with the other case, suppose $C \neq \emptyset$. Let $c \in C$.

Thus $(g \circ f)(a) = c$. Therefore $g \circ f$ is surjective. □

2. Let A, B, and C be sets. Suppose $f : A \to B$ and $g : B \to C$ are both injective. Finish proving that $g \circ f : A \to C$ is injective.

Proof. Let A, B, and C be sets. Suppose $f : A \to B$ and $g : B \to C$ are both injective. If $A = \emptyset$, then $g \circ f$ is trivially injective. To deal with the other case, suppose $A \neq \emptyset$. Let $a_1, a_2 \in A$ and suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$.

Thus $a_1 = a_2$. Therefore $g \circ f$ is injective. □

Definition. If $f : A \to B$ is both injective and surjective, then it is called a **bijection**.

If there is a bijection between sets A and B, then the sets must be the same size. For example, Let $A = \{a, b, c\}$ and let $B = \{1, 2, 3\}$. It is easy to find bijection between A and B: $\{(a, 1), (b, 2), (c, 3)\}$. This turns out to give a good definition of the size (or cardinality) of a set.

Definition. Let A and B be sets. Then we say A and B have the same **cardinality** and write $|A| = |B|$ if there is a bijection between A and B.

The definition matches our expectations for finite sets. Things get a little stranger when dealing with infinite sets.

3. Let $A = \{2n : n \in \mathbb{Z}\}$ and $B = \{2n + 1 : n \in \mathbb{Z}\}$. For each of the following statements find a bijection that proves the statement is true:

a) $|A| = |B|$

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b) $|A| = |\mathbb{Z}|$

c) $|\mathbb{N}| = |\mathbb{Z}|$

Theorem (Shröder-Bernstein)**.** *Let A and B be sets. If there is an injective function from A to B and another injective function from B to A, then there is a bijection between A and B.*

4. We can use the Shröder-Bernstein theorem to prove that $|\mathbb{Z}| = |\mathbb{Q}|$. We just need to:

a) Produce an injective function from $\mathbb{Z}$ to $\mathbb{Q}$.

b) Produce an injective function from $\mathbb{Q}$ to $\mathbb{Z}$.

5. Prove that $|\mathbb{N}| = |\mathbb{Q}|$.

6. Do you think all infinite sets have the same cardinality?