Name(s):

Math 301 Problem Proofs February 20. 2015

Instructions: Correct the following proofs. Keep in mind the guidelines of section 5.3.

Proposition 1. *Let $a \in \mathbb{Z}$. Prove that if a is odd, then $a+1$ is even.*

Proof. By definition, if a is *odd*, then $a = 2x+1$. Thus $a+1 = (2x+1)+1 = 2x+2 = 2(x+1)$. By definition, if a is *even*, then there is an integer y such that $a = 2y$. Therefore, $a+1$ is even. □

Proposition 2. *If 7 does not divide ab, then 7 divides neither a nor b.*

Contrapositive Proof.

Proof. Let $a, b \in \mathbb{Z}$. Suppose that 7 divides a or 7 divides b.

Case 1. Without loss of generality, suppose 7 divides a but 7 does not divide b. By definition $a = 7x$ for some $x \in \mathbb{Z}$ and $b \neq 7y$ for some $y \in \mathbb{Z}$. Hence $ab = 7(xb)$. Therefore 7 divides ab.

Case 2. Suppose 7 divides a and 7 divides b. By definition $a = 7m$ and $b = 7n$ for some $m, n \in \mathbb{Z}$. Then $ab = 7(7mn)$. Therefore 7 divides ab. □

Proposition 3. *Let $a, b \in Z$. Prove that if 7 does not divide ab, then 7 divides neither a nor b.*

Proof. (Contrapositive) Suppose 7 divides a or 7 divides b and $a, b \in \mathbb{Z}$. By definition, $7m = a$ and $7n = b$ for some $m, n \in Z$. Thus, if 7 divides a, then $ab = 7(mb)$. Next, if 7 divides b, then $ab = 7(na)$. In either situation, ab is a multiple of seven. Therefore, it follows that 7 divides ab. Since the contrapositive is true, it follows that the original statement is true. □

Proposition 4. *Let $x \in \mathbb{R}$. If $x^2 + 5x < 0$, then $x < 0$.*

Contrapositive: Let $x \in \mathbb{R}$. If $x \geq 0$, then $x^2 + 5x \geq 0$.

Proof. Let x be a real number greater than or equal to 0.

Case 1: $x > 0$. By definition of positive numbers, since $x > 0$, $5x > 0$. By definition of squares, x^2 is greater than 0. Since two positive numbers added together equal a third positive number, $x^2 + 5x$ is greater than 0.

Case 2: $x = 0$. Since a number times 0 is 0, $5x = 5(0) = 0$. Similarly, $0^2 = 0$. Thus, $x^2 + 5x = 0 + 0 = 0$.

Therefore, for all real numbers $x \geq 0$, $x^2 + 5x \geq 0$.

Therefore, by contrapositive, if $x^2 + 5x < 0$, then $x < 0$. □

Proposition 5. *Let $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|(b + c)$, then $a|c$*

Proof. Let $a, b, c \in \mathbb{Z}$ and suppose that $a|b$ and $a|(b+c)$. By definition, there exists $x, y \in \mathbb{Z}$ such that $ax = b$ and $ay = b+c$. Then $ay - ax = a(y - x) = b + c - b = c$. Therefore, since $y - x$ is an integer, $a|c$. □

Proposition 6. *For any $a, b \in \mathbb{Z}$, it follows that $(a + b)^3 \equiv a^3 + b^3 (mod 3)$.*

Proof. Suppose $a, b \in \mathbb{Z}$. Thus, there is an integer x such that $x = a^2b + ab^2$. Hence $3x = 3a^2b + 3ab^2 = (a + b)^3 - a^3 - b^3$. It follows that $3|(a + b)^3 - a^3 - b^3$. Therefore, $(a + b)^3 \equiv a^3 + b^3 (mod 3)$. □