NAME(S): SOLUTIONS

MATH 301 FIRST PROOFS FEBRUARY 6, 2015

Definition. Let $n, m \in \mathbb{Z}$.

1. n is *even* if there is $a \in \mathbb{Z}$ such that $n = 2a$.
2. n is *odd* if there is $a \in \mathbb{Z}$ such that $n = 2a + 1$.
3. m *divides* n (written $m|n$) if there is $a \in \mathbb{Z}$ such that $ma = n$.
4. If $n \geq 2$ and the only positive divisors n are 1 and n, then n is *prime*.
5. If $n \geq 2$ and n is not prime, then n is *composite*.

1. Let $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|c$, then $a|(b + c)$.

Proof. Let $a, b, c \in \mathbb{Z}$ and suppose that $a|b$ and $a|c$. By definition there are integers x and y such that $ax = b$ and $ay = c$. It follows that $b + c = ax + ay = a(x + y)$. Therefore by definition $a|(b + c)$.

□

2. Let $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|(b + c)$, then $a|c$.

Proof. Let $a, b, c \in \mathbb{Z}$ and suppose that $a|b$ and $a|(b + c)$. By definition there are integers x and y such that $ax = b$ and $ay = b + c$. Consequently $c = (b + c) - b = ay - ax = a(y - x)$. Therefore by definition $a|c$.

□

3. Let $x, y \in \mathbb{Z}$.

a) If x or y is even, then xy is even.

Proof. Let $x, y \in \mathbb{Z}$. Without loss of generality, suppose x is even. Then by definition there is an integer a such that $x = 2a$. Consequently $xy = 2ay$. Therefore xy is even.

□

b) State the converse of "if x or y is even, then xy is even". Note that the converse has the same implicit quantifier(s) as the original statement.

If xy is even, then x or y is even.

c) Is the converse true or false? (There is no need to prove it).

The converse is true (because 2 is prime).

d) It is also true that if $6|x$ or $6|y$, then $6|(xy)$. Is the converse of this statement true?

The converse of the statement is "ff $6|(xy)$, then $6|x$ or $6|y$". In this case the converse is false because there are integers x and y for which $6|xy$ but $6 \nmid x$ and $6 \nmid y$.

4. Every odd integer is the difference of two squares (e.g. $5 = 3^2 - 2^2$).

Proof. Suppose x is an odd integer. Then by definition there is an integer a such that $x = 2a + 1$. It then follows that $x = (a + 1)^2 - a^2$. Therefore x is the difference of two squares.

□