Determine if each statement is true or false. If it is true, prove it. If it is false, give a disproof.

1. There is an integer n such that $n^2 \equiv -1 \pmod{13}$.

Solution. The statement is true. An example suffices to prove that such an integer exists: $2(13) = 26 = 25 - (-1)$. Hence $5^2 \equiv -1 \pmod{13}$.

2. For all $a, b, c \in \mathbb{Z}$, if $a|bc$, then $a|b$ or $a|c$.

Solution. The statement is false. A counterexample suffices to prove the negation: there are $a, b, c \in \mathbb{Z}$ such that $a|bc$ but $a \nmid b$ and $a \nmid c$. Let $a = 6$, $b = 2$, and $c = 3$. We see that $6|6$ (so $a|bc$) but $6 \nmid 2$ (so $a \nmid b$) and $6 \nmid 3$ (so $a \nmid c$).

3. There are integers a and b such that $12a + 15b = 2$.

Solution. The statement is false. A disproof entails proving that for any integers a and b it follows that $12a + 15b \neq 2$. We proceed by contradiction. Suppose a and b are integers such that $12a + 15b = 2$. It follows that $3(4a + 5b) = 2$ and thus that $3|2$. This is a contradiction.

4. There are integers a and b such that $11a + 15b = 1$.

Solution. The statement is true. An example suffices to prove it: $11(-4) + 15(3) = 1$.

5. If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

Solution. The statement is false. A counterexample suffices to prove the negation: there are sets X, A, and B such that $X \subseteq A \cup B$ but $X \nsubseteq A$ and $X \nsubseteq B$. Let $X = \{1, 2\}$, $A = \{1\}$, and $B = \{2\}$. Then $A \cup B = \{1, 2\} \supseteq X$ but $X \nsubseteq A$ and $X \nsupseteq B$.

Challenge. There is an integer n such that $n^2 \equiv -1 \pmod{3}$.

Solution. This statement is false. A disproof entails proving that for any integer n it follows that $n^2 \not\equiv -1 \pmod{3}$. The proof I have in mind is direct and involves three cases: $n \equiv 0 \pmod{3}$, $n \equiv 1 \pmod{3}$, and $n \equiv 2 \pmod{3}$. We can then determine that in the first case $n^2 \equiv 0 \pmod{3}$ and in the latter two cases $n^2 \equiv 1 \pmod{3}$. In none of these cases is $n^2 \equiv -1 \pmod{3}$.