

1. Find the flaw(s) in the following induction proof.

Proposition. *All dogs are the same color.*

Proof by induction. The proposition may be interpreted as saying that every set of n dogs contains dogs of only one color. We proceed by induction on the number n .

Base case: $n = 1$. This means that we have a set of just one dog. Obviously, if there's just one dog, then all dogs in the set are the same color.

Inductive step: let $k \in \mathbb{N}$ and suppose that any set of k dogs consists of dogs of just one color. Let A be a set of $k + 1$ dogs. Let a be one of the dogs in A . Then the set $A - \{a\}$ is a set of k dogs and hence they are all the same color (by the inductive hypothesis). Let b be another dog in A . Similarly, $A - \{b\}$ is a set of k dogs and thus all of these dogs are the same color. Because $a \in A - \{b\}$ and $b \in A - \{a\}$ it follows that dog a and dog b are the same color as all the other dogs in A . Hence A consists of dogs of only one color.

Therefore, by induction, all dogs are the same color. □

Solution. The problem here is that the inductive step works only for a set A with cardinality 3 or greater. In fact, it is not true that every set of two dogs contains dogs of only one color, so it makes sense that this does not follow from the base case.

2. Use induction to prove that for any non-negative integer n , any set with n elements has 2^n subsets.

Proof by induction. Base case: $n = 0$. The only set with zero elements is \emptyset . The only subsets of \emptyset is \emptyset . Hence every subset with zero elements has $2^0 = 1$ subset.

Now suppose that k is a non-negative integer and every set with k elements has 2^k subsets. Let A be a set with $k + 1$ elements. Let $a \in A$ (we know that A has at least one element because $k + 1 \geq 1$). Then $A - \{a\}$ is a set with k elements and by hypothesis this set has 2^k subsets. These are exactly the subsets of A that do not contain a . It remains to count the number of subsets of A that do contain a . Each such subset is a subset of $A - \{a\}$ with the element a added in. Thus there are 2^k subsets of A that contain the element a . Therefore A has $2^k + 2^k = 2^{k+1}$ subsets in total.

Therefore, by induction, a set with n elements has 2^n subsets. □

3. Prove that for any $n \in \mathbb{N}$, $1(2) + 2(3) + 3(4) + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Proof by induction. Base case: $n = 1$. Then $1(2) = \frac{1(2)(3)}{3}$, so the base case holds.

Now let $k \in \mathbb{N}$ and suppose that $1(2) + 2(3) + 3(4) + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$. Then

$$\begin{aligned} 1(2) + 2(3) + 3(4) + \cdots + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}$$

Therefore, by induction, $1(2) + 2(3) + 3(4) + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for every $n \in \mathbb{N}$. □

4. Prove that for any $n \in \mathbb{N}$, $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

Proof by induction. Base case: $n = 1$. Then $\frac{1}{1} = 2 - \frac{1}{1}$, so the base case holds.

Now let $k \in \mathbb{N}$ and suppose that $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$. Then

$$\begin{aligned} \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\ &= 2 - \frac{(k+1)^2 - k}{k(k+1)^2} \\ &= 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\ &= 2 - \frac{k(k+1)}{k(k+1)^2} - \frac{1}{k(k+1)^2} \\ &\leq 2 - \frac{k(k+1)}{k(k+1)^2} \\ &= 2 - \frac{1}{k+1}. \end{aligned}$$

Therefore, by induction, $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$. □