Definition 1. A relation R on a set A is an *equivalence relation* if it is reflexive ($\forall a \in A,\ aRa$), symmetric ($\forall a, b \in A,\ aRb \implies bRa$), and transitive ($\forall a, b, c \in A,\ (aRb \wedge bRc) \implies aRc$). If R is an equivalence relation on A and $a \in A$, then the *equivalence class of* a is

$$[a] = \{x \in A : xRa\}.$$

Example. We have shown that congruence modulo 3 is an equivalence relation on $\mathbb{Z}$. This equivalence relation has exactly 3 equivalence classes:

- $[0] = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$
- $[1] = \{\dots, -8, -5, -2, 1, 4, 7, 10, \dots\}$
- $[2] = \{\dots, -7, -4, -1, 2, 5, 8, 11, \dots\}$

1. What are the equivalence classes for congruence modulo 7 (as a relation on $\mathbb{Z}$)?

2. Consider the following relation on $\mathbb{Z}$: $R = \big\{(x, y) : x^2 \equiv y^2 \pmod 7\big\}$.

a) Prove that R is an equivalence relation.

b) Determine the equivalence classes for this relation. Hint: there are 4 equivalence classes.

Definition 2. A *partition* of the set A is a collection of nonempty subsets of A such that both of the following hold:

1. the union of all the subsets is A;
2. the intersection of any two of the subsets is $\emptyset$.

3. Note that the equivalence classes in the example of congruence modulo 3 form a partition of $\mathbb{Z}$. Your equivalence classes for congruence modulo 7 and R should also form partitions of $\mathbb{Z}$. Verify that this is the case.

4. We proved that for any $n \in \mathbb{N}$, the relation $\equiv \pmod{n}$ is an equivalence relation on $\mathbb{Z}$. The following questions deal with this relation with n left as an arbitrary number.

a) What are the equivalence classes of this relation? How many are there?

b) Let $a, b \in \mathbb{Z}$. Prove that for any $x \in [a]$ and any $y \in [b]$, it follows that $[x + y] = [a + b]$.

c) Under the same hypotheses as c, does it follow that $[xy] = [ab]$? Prove you are right.

5. Find the equivalence class of $17(134 - 51)$ for the relation $\equiv \pmod{7}$ (this should be easy to do without a calculator if you use parts b and c of problem 4).

6. Find the units digits in the standard decimal expansions of 496^3 and 4^{41}.