1. Prove that if each point in the plane $\mathbb{R}^2$ is colored either red or blue, then there must be two points of the same color exactly one unit apart.

Solution. Consider the corners of an equilateral triangle with sides of length 1. Each of these 3 points is colored either red or blue, hence two of them have the same color. Those two points are exactly one unit apart. Therefore there must exist at least two points of the same color exactly one unit apart.

2. In each square of a 5×5 board there is a flea. At some point, all the fleas jump to an adjacent square (two squares are adjacent if they share an edge). Is it possible that after they settle in the new squares, the configuration is exactly as before: one flea per square? Hint: suppose the board is colored like a chess board; how many squares of each color are there?

Solution. Suppose each square of a 5×5 board is occupied by a flea. Color the board like a chessboard. This means that there are 13 squares of one color, which we without loss of generality may assume is black, and 12 squares of the other color, white. Suppose that each flea jumps to an adjacent square. By moving to an adjacent square each flea moves to a square with color different from the color of its starting square. Thus the 13 fleas on black squares must move to the 12 white squares. Hence more than one of these fleas must land in the same square. Therefore the configuration of the fleas is different after they move.

Definition 1. Let A, B, and C be sets and let $f : A \to B$ and $g : B \to C$ be functions. The composite function $g \circ f : A \to C$ is defined by $(g \circ f)(x) = g(f(x))$.

3. Prove that if f and g are both injective, then the composite function $g \circ f$ is injective.

Solution. Suppose $f : A \to B$ and $g : B \to C$ are both injective and that $a, b \in A$ with $a \neq b$. Because f is injective it follows that $f(a) \neq f(b)$. Because g is injective it then follows that $g(f(a)) \neq g(f(b))$. Thus $(g \circ f)(a) \neq (g \circ f)(b)$. Therefore $g \circ f$ is injective.

4. Prove that if f and g are both surjective, then the composite function $g \circ f$ is surjective.

Solution. Suppose $f : A \to B$ and $g : B \to C$ are both surjective and let $c \in C$. Because g is surjective there is $b \in B$ such that $g(b) = c$. Because f is surjective there is $a \in A$ such that $f(a) = b$. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$. Therefore $g \circ f$ is surjective.

5. Writing as little as possible, prove that if f and g are both bijective, then $g \circ f$ is bijective.

Solution. Suppose $f : A \to B$ and $g : B \to C$ are both bijections. Then by definition f and g are both injective and both surjective. B the preceding problems the composition $g \circ f$ is both invective and surjective. Therefore $g \circ f$ is a bijection.