Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 2$, then $\sqrt[n]{2}$ is irrational.*

Proof. Let $n \in \mathbb{N}$. Suppose, by contrapositive, that if $\sqrt[n]{2}$ is rational, then $n < 2$.
By simplifying, $\sqrt[n]{2} = 2^{\frac{1}{n}}$.
Then, by definition, $2^{\frac{1}{n}}$ is rational if $a, b \in \mathbb{Z}$ where $\frac{a}{b} = 2^{\frac{1}{n}}$ such that $b \neq 0$.
Observe that $2^{\frac{1}{n}}$ can only be rational when $n < 2$ where $n \in \mathbb{N}$. Thus, $n = 1$.
When $n = 1$, $2^{\frac{1}{n}} = \frac{a}{b}$ is true.
Therefore, if $n \geq 2$, then $\sqrt[n]{2}$ is irrational.

Comments:

Proposition 2. *If $a \in \mathbb{Z}$, then $4 \nmid (a^2 - 3)$.*

Proof. Let $a \in \mathbb{Z}$.

Case 1: Suppose a is even. By definition, there is $a \in \mathbb{Z}$ such that $a = 2c$. We can see that $4|((2c)^2 - 3) = 4|(4c^2 - 3)$. This shows us that 4 does not divide into $(a^2 - 3)$ when a is even.

Case 2: Suppose a is odd. By definition, there is $a \in \mathbb{Z}$ such that $a = 2c - 1$. We can see that $4|((2c-1)^2 - 3) = 4|(4c^2 - 4c - 2) = 4|2(2c^2 - 2c - 1)$. This shows us that 4 does not divide into $(a^2 - 3)$ when a is odd. Therefore, we can see that $4|(a^2 - 3)$ (but it actually doesn't divide).

Comments:

Proposition 2. *If $a \in \mathbb{Z}$, then $4 \nmid (a^2 - 3)$*

Proof. Let $a \in \mathbb{Z}$.

Case 1. a is even.
By definition, $\exists n \in \mathbb{Z}$ such that $a = 2n$.
Now, $a^2 - 3 = (2n)^2 - 3$.

$$\begin{aligned} a^2 - 3 &= (2n)^2 - 3 \\ &= 4n^2 - 3 \\ &= 2(2n^2 - 2) + 1 \end{aligned} \tag{1}$$

$2(2n^2 - 2) + 1$ is odd, so therefore $4 \nmid (a^2 - 3)$.

□

Case 2. a is odd.
By definition, $\exists n \in \mathbb{Z}$ such that $a = 2n + 1$. Now, $a^2 - 3 = (2n + 1)^2 - 3$.

$$\begin{aligned} a^2 - 3 &= (2n+1)^2 - 3 \\ &= 4n^2 + 4n + 1 - 3 \\ &= 4n^2 + 4n - 2 \\ &= 2(2n^2 + 2n - 1) \end{aligned} \tag{2}$$

$2n^2 + 2n - 1 = 2(n^2 + n - 1) + 1$, which is odd. Hence, we cannot factor another 2 out of $2n^2 + 2n - 1$.
Since $2 \nmid (2n^2 + 2n - 1)$, we know that $4 \nmid [2(2n^2 + 2n - 1)]$, which is the same as saying that $4 \nmid (a^2 - 3)$. □

In both cases, $4 \nmid (a^2 - 3)$. □

Comments:

Proposition 2. *If $a \in \mathbb{Z}$, then $4 \nmid (a^2 - 3)$.*

Proof. Let $a \in \mathbb{Z}$.

Case 1: a is even.
By definition of even, there exists $c \in \mathbb{Z}$ such that $a = 2c$.
Then $a^2 - 3 = 2a^2 - 3 \rightarrow 4(a^2 - 1) + 1$.
So $a^2 - 3 = 4y + 1$ where $y = a^2 - 1$

By definition of congruence, there exists $a, b \in \mathbb{Z}$ where $a \equiv b(modn)$ if $n|(a - b)$.
More precisely $n|(a - b)$ can be put as $a = nk + b$ where $k \in \mathbb{Z}$.
Since $a^2 - 3 = 4y + 1$, we can say $a^2 - 3 \equiv 1(mod4)$.
Thus, $\not| a^2 - 3$.

Case 2: a is odd.
By definition of odd, there exists $c \in \mathbb{Z}$ such that $a = 2c + 1$.
Then $a^2 - 3 = (2c + 1)^2 - 3 \rightarrow 4c^2 + 4c - 2 \rightarrow 4(c^2 + c - 1) + 2$.
So $a^2 - 3 = 4(y) + 2$ where $y = c^2 + c - 1$.

By definition of congruence, since $a^2 - 3 = 4y + 2$, we can say $a^2 - 3 \equiv 2(mod4)$
Thus, $4 \not| (a^2 - 3)$.
Therefore, if $a \in \mathbb{Z}$, then $4 \not| (a^2 - 3)$.

□

Comments:

Proposition 2. *If* $a \in \mathbb{Z}$, *then* $4 \nmid (a^2 - 3)$.

Using the negation of the proposition yields: $a \in \mathbb{Z}$ and $4 \mid (a^2 - 3)$

Proof: Let $a \in \mathbb{Z}$ and suppose $4 \mid (a^2 - 3)$. By definition (of divides), there is $m \in \mathbb{Z}$ such that $4m = a^2 - 3$. WLOG, suppose a is even. By definition (of even), there is $k \in \mathbb{Z}$ such that $a = 2k$.

$$4m = (2k)^2 - 3$$
$$4(k^2 - m) = 3$$
$$k^2 - m = \frac{3}{4}$$

Since $k^2 \in \mathbb{Z}$ and $\frac{3}{4} \in \mathbb{Q}$, then $k^2 - \frac{3}{4} \notin \mathbb{Z}$. This shows that $n \notin \mathbb{Z}$, which is a contradiction because this goes against the assumption that $n \in \mathbb{Z}$. Thus, $4 \nmid (a^2 - 3)$. □

Comments:

Proposition 2. *If* $a \in \mathbb{Z}$, *then* $4 \nmid (a^2 - 3)$

Proof. Let $a \in \mathbb{Z}$. a can either be even or odd.

Case 1. a is even.
By definition, there exists $n \in \mathbb{Z}$, such that $a = 2n$.
Then $a^2 - 3 = (2n)^2 - 3 \to a^2 - 3 = 4n^2 - 3 \to a^2 - 3 = 4(n^2 - 3/4)$.
Thus $4 \nmid (a^2 - 3)$, because $4 \nmid 3$.

Case 2. a is odd.
By definition, there exists $k \in \mathbb{Z}$, such that $a = 2k + 1$.
Then $a^2 - 3 = (2k + 1)^2 - 3 \to a^2 - 3 = 4k^2 + 4k - 2 \to a^2 - 3 = 4(2k^2 + 2k - 1/2)$.
Thus $4 \nmid (a^2 - 3)$, because $4 \nmid -2$.

Therefore in all cases $4 \nmid (a^2 - 3)$. □

Comments:

Definition 1. Let $a, b \in \mathbb{Z}$.

a) a is **even** if there is $c \in \mathbb{Z}$ such that $a = 2c$. (This is the same as $2|a$).
b) a is **odd** if there is $c \in \mathbb{Z}$ such that $a = 2c + 1$.
c) a **divides** b, written $a|b$, if there is $c \in \mathbb{Z}$ such that $ac = b$. (Also expressed "b is divisible by a").

Proposition 2. *If* $a \in \mathbb{Z}$*, then* $4 \nmid (a^2 - 3)$

Proof. Let $a \in \mathbb{Z}$. If $a^2 - 3$ has a remainder when divided by 4, then $4 \nmid (a^2 - 3)$.

Case 3. Suppose a is even. By definition(a) $a = 2c$ where $c \in \mathbb{Z}$.

$$\begin{aligned} a &= 2c \\ a^2 &= (2c)^2 \\ a^2 - 3 &= (2c)^2 - 3 \\ &= 4c^2 - 3 \\ (4c^2 - 3) mod(4) &= 1 \end{aligned}$$

Case 4. Suppose a is odd. By definition(b) $a = 2c + 1$ where $c \in \mathbb{Z}$.

$$\begin{aligned} a &= c + 1 \\ a^2 &= (2c + 1)^2 \\ a^2 - 3 &= (2c + 1)^2 - 3 \\ &= 4c^2 + 4c + 1 - 3 \\ & 4c^2 + 4c - 2 \\ (4c^2 + 4c - 2) mod(4) &= 2 \end{aligned}$$

Thus if a is even and $(4c^2 - 3) mod(4) \neq 0$ or if a is odd and $(4c^2 + 4c - 2) mod(4) \neq 0$, then $4 \nmid (a^2 - 3)$ because there is a remainder when divided by 4. □

Comments:

Proposition 2. *If* $a \in \mathbb{Z}$*, then* $4 \nmid (a^2 - 3)$

Proof:
Let a $\in \mathbb{Z}$. Suppose $a^2 - 3$ is odd or even.

Case 1:
Let a be an odd integer.
By definition of odds, $a = 2m + 1$ such that m $\in \mathbb{Z}$
Then $a^2 - 3 = (2m + 1)^2 - 3 = 4m^2 + 4m + 1 - 3 = 4m^2 + 4m + 2$
Thus, $4m^2 - 3$ is not divisible by 4

Case 2:
Let a be an even integer
By definition of even, $a = 2m$ such that m $\in \mathbb{Z}$.
Then $a^2 - 3 = (2m)^2 - 3 = 4m^2 - 3$
Thus $4m - 3$ is not divisible by 4.

Therefore, 4 is not divisible by $(a^2 - 3)$

Comments:

Proposition 2. *If* $a \in \mathbb{Z}$*, then* $4 \nmid (a^2 - 3)$

Proof. Let $a \in \mathbb{Z}$ and suppose $4 \mid (a^2 - 3)$. By definition (of divides) there exists an integer x such that $4x = a^2 - 3$. It follows that $4x = 2(2x) = a^2 - 3$. Thus $a^2 - 3$ must be an even integer since $2 \mid (a^2 - 3)$. Suppose $a = 4$. Then $a^2 - 3 = 4^2 - 3 = 16 - 3 = 13$. This is a contradiction since 13 is not even. Thus $2 \nmid (a^2 - 3)$ for some integers. Therefore $4 \nmid (a^2 - 3)$. □

Comments:

Proposition 2. *If* $a \in \mathbb{Z}$*, then* $4 \nmid (a^2 - 3)$

Proof. (By Contradiction) Suppose $(a^3 - 3) \in \mathbb{Z}$ and $4 \mid (a^3 - 3)$
By definition $4m = a^3 - 3$ where $m \in \mathbb{Z}$
Case 1:Let a be even
Hence a=2k where $k \in \mathbb{Z}$
Then, $4m = (2k)^2 - 3 \rightarrow 4m = 4k^2 - 3 \rightarrow m = k^2 - \frac{3}{4}$
By Definition, an $\mathbb{Z} + \mathbb{Z} = \mathbb{Z}$
Since $k^2 \in \mathbb{Z}$ and $\frac{3}{4}$ is not, this is a contradiction of the original statement, which said m has to be an integer.
Case 2: Let a be odd
Hence, a=2k+1 where $k \in \mathbb{Z}$
Then, $4m = (2k+1)^2 - 3 \rightarrow 4m = 4k^2 + 4k - 2 \rightarrow m = k^2 + k - \frac{1}{2}$
By Definition, an $\mathbb{Z} + \mathbb{Z} = \mathbb{Z}$
Since $k^2 + k \in \mathbb{Z}$,but $\frac{1}{2}$ is not, this contradicts the previous statement that $m \in \mathbb{Z}$
Therefore, if $a \in \mathbb{Z}$ then, $4 \nmid (a^3 - 3)$

□

Comments:

Proposition 2. *If* $a \in \mathbb{Z}$*, then* $4 \nmid (a^2 - 3)$

Proof. Suppose $a \in \mathbb{Z}$ and $4|(a^2 - 3)$.

By definition (of divides), there is $c \in \mathbb{Z}$ such that $4c = a^2 - 3$.

Case 1: a is odd. By definition, there is $m \in \mathbb{Z}$ such that $a = 2m + 1$. Then, $4c = (2m+1)^2 - 3 = 4m^2 + 4m + 1 - 3 = 4m^2 + 4m - 2 \implies 4c = 2(2m^2 + 2m - 1)$ and $2m^2 + 2m - 1 \in \mathbb{Z}$. Thus, $c \notin \mathbb{Z}$ since 4 does not divide 2.

Case 2: a is even. By definition, there is $n \in \mathbb{Z}$ such that $a = 2n$. Then, $4c = (2n)^2 - 3 = 4n^2 - 3 = 4n^2 - 4 + 1 \implies 4c = 2(2n^2 - 2) + 1$ and $2n^2 - 2 \in \mathbb{Z}$. Thus, $c \notin \mathbb{Z}$ since 4 does not divide 2 and 4 does not divide 1.

Both cases contradict $4|(a^2 - 3)$.

Therefore, $4 \nmid (a^2 - 3)$.

□

Comments:

Proposition 3. *For any natural numbers a and b, $a = lcm(a, b)$ if and only if $b \mid a$.*

Notice that $lcm(a, b) * gcd(a, b) = a * b$.
Since $lcm(a, b) = a$, then $a * gcd(a, b) = a * b$.
Thus, $gcd(a, b) = b$ if and only if $b|a$.
Therefore, $lcm(a, b) = a$ if and only if $b|a$.

Comments:

Proposition 3. *For any natural numbers a and b, $a = lcm(a, b)$ if and only if $b \mid a$.*

Proof. Part 1: Let $a, b \in \mathbb{N}$ and suppose $a = lcm(a, b)$. By definition, a is the smallest positive integer where $a|a$ and $b|a$. Therefore, by definition, $b|a$.

Part 2: Let $a, b \in \mathbb{N}$ and suppose $b|a$. By definition, there exists a $c \in \mathbb{Z}$ such that $bc = a$. Hence b is less than or equal to a. Also, numbers always divide themselves, so we know $a|a$. Since both b and a divide a, we know a is a common multiple between the two. As the smallest multiple (greater than 0) of any number is itself, a is the smallest multiple of itself. Therefore, $a = lcm(a, b)$.

As it can be shown that if given one condition we can prove the other, $a = lcm(a, b)$ if and only if $b|a$.

□

Comments:

Proposition 4. *Let C be a circle in $\mathbb{R}^2$ centered at $(1,1)$. Then either $(2,3) \notin C$ or $(0,2) \notin C$. (Note that C is just the circle itself, not the interior).*

Proof. Let C be a circle in $\mathbb{R}^2$ centered at $(1,1)$.

Assume $(2,3) \in C$. Then the distance from the center $(1,1)$ is $\sqrt[2]{(2-1)^2+(3-1)^2} = \sqrt[2]{1+4} = \sqrt[2]{5}$.

Now assume $(0,2) \in C$. Then the distance from the center $(1,1)$ is $\sqrt[2]{(0-1)^2+(2-1)^2} = \sqrt[2]{1+1} = \sqrt[2]{2}$

Since the radius of C can not have both a radius of $5^(1/2)$ and of $2^(1/2)$, C can not contain both $(2,3)$ and $(0,2)$. Therefore, either $(2,3) \notin C$ or $(0,2) \notin C$. $\square$

Comments: