Proposition 1. *Any two successive Fibonacci numbers are relatively prime.*

Proof. (By Mathematical Induction):

Base Case (n=1): For $n = 1$, we see that $F_3 = 2$ and $F_{1+1} + F_1 = 1 + 1$, so $2 = 2$, and $gcd(F_1, F2 = 1$. Thus it is true for $n = 1$.

Inductive Step: Now let $k \in \mathbb{Z}$ and assume that $F_k and F_{k+1}$ are relatively prime for some integer $k \geq 1$, since $F_{k+2} = F_{k+1} + F_k$. If d is a positive common divisor of F_{k+1} and F_{k+2}, then $d|F_k$, also so that d is a positive common divisor of F_k and F_{k+1}. By the inductive hypothesis, F_k and F_{k+1} are relatively prime, so that d=1. Therefore, F_{k+1} and F_{k+2} are also relatively prime.

By the Principle of Mathematical Induction, the proposition holds that two consecutive Fibonacci numbers are relatively prime.

□

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$.

Proof. **Base Case** $(n = 1)$**:** $(1)^2 = 1 = 1^3$, so the proposition is true for $n = 1$.

Inductive step: Let $k \in \mathbb{N}$ and suppose that $(1 + 2 + 3 + ... + k)^2 = 1^3 + 2^3 + 3^3 + ... + k^3$. Therefore, $(1 + 2 + 3 + ... + k + (k + 1))^2 = (1 + 2 + 3 + ... + k)^2 + 2(k + 1)(1 + 2 + 3 + ... + k) + (k + 1)^2$ Hence, $(1 + 2 + 3 + ... + k + (k + 1))^2 = 1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3$.

Therefore, using induction, the proposition is true.

□

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Proof. Observe that $(1+2+3+\ldots+n) = \frac{n(n+1)}{2}$. Thus, $\frac{(n(n+1))^2}{2^2} = 1^3+2^3+3^3+\ldots+n^3$.

Let $n=1$. Then, $[\frac{1(1+1)}{2}]^2 = (1)^3$. So, $[\frac{2}{2}]^2 = 1$. Which is true.

Assume the above results are true for all $n=k$. Thus,

$$1^3+2^3+3^3+\ldots+k^3 = [\frac{k(k+1)}{2}]^2.$$

Now, the results must be proved for $n=k+1$. So,

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = [\frac{(k+1)(k+2)}{2}]^2.$$

Consider,

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = [\frac{k(k+1)}{2}]^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{2^2} + (k+1)^3 = (k+1)^2[\frac{k^2}{4}+k+1] = (k+1)^2[\frac{k^2+4k+4}{4}]$$

$$= \frac{(k+1)^2(k+2)^2}{2^2} = [\frac{(k+1)(k+2)}{2}]^2.$$

Hence, the results are true for $n = \frac{R.H.S.}{k+1}$. Therefore by induction, the initial statement is true for all $n \in \mathbb{N}$. □

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Proof. Proof by Induction.

Base Case: $n = 1$. Then the statement simplifies to $1^2 = 1^3$, so $1 = 1$. Therefore, $n = 1$ is true.

Let $n \in \mathbb{N}$, and suppose $(1+2+3+...+n)^2 = 1^3+2^3+3^3+...+n^3$.

Because S_n is true, prove $S_n + 1$ given $(1+2+3+...+n)^2 = 1^3+2^3+3^3+...+n^3$.

Prove $(1+2+3+...+n+(n+1))^2 = (1^3+2^3+3^3+...+n^3+(n+1)^3)$.

Expanding the equation in the form $(a+b)^2 = a^2+2ab+b^2$, we get

$(1+2+3+...+n)^2 + 2(1+2+3+...+n)(n+1) + (n+1)^2$.

Since $(1+2+3+...+n)^2 = 1^3+2^3+3^3+...+n^3$, we can substitute.

So, $(1+2+...+n+(n+1))^2 =$

$(1^3+2^3+3^3+...+n^3) + 2(1+2+3+...+n)(n+1) + (n+1)^2$.

Recognize that $(1+2+3+...+n) = (n(n+1))/2$.

By substitution we get $(1+2+...+n+(n+1))^2 =$

$(1^3+2^3+...+n^3) + (2n(n+1))/2 + (n+1)^2$.

Thus, $(1+2+...+n+(n+1))^2 =$

$(1^3+2^3+...+n^3) + (n(n+1)^2) + (n+1)^2 =$

$(1^3+2^3+...+n^3) + n^3+3n^2+3n+1 =$

$(1^3+2^3+...+n^3) + (n+1)(n+1)^2 =$

$(1^3+2^3+...+n^3) + (n+1)^3$.

Therefore, $(1+2+3+...+n+(n+1))^2 = (1^3+2^3+3^3+...+n^3+(n+1)^3)$. □

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Proof. Let $n \in \mathbb{N}$
Base Case (n = 1): Observe that $(1)^2 = 1^3$, which is true.

Induction Case:
Let $n = k$, where $k \in \mathbb{N}$, and suppose that $(1+2+3+\cdots+k)^2 = 1^3+2^3+3^3+\cdots+k^3$ is true.
Consider $n = k+1$.
Then, $(1+2+3+\cdots+k+(k+1))^2 = 1^3+2^3+3^3+\cdots+k^3+(k+1)^3$
We know that: $1+2+\cdots+k = \frac{k(k+1)}{2}$ by Gauss's formula.

$$
\begin{aligned}
&= ((1+2+3+\cdots+k)+(k+1))^2 \\
&= (1+2+3+\cdots+k)^2+(k+1)^2+2(1+2+3+\cdots+k)(k+1) \\
&= (1^3+2^3+3^3+\cdots+k^3)+(k+1)^2+\frac{2k(k+1)}{2}(k+1) \\
&= (1^3+2^3+3^3+\cdots+k^3)+(k+1)^2+k(k+1)^2 \\
&= (1^3+2^3+3^3+\cdots+k^3)+k^2+2k+1+k^3+2k^2+k \\
&= (1^3+2^3+3^3+\cdots+k^3)+k^3+3k^2+3k+1 \\
&= (1^3+2^3+3^3+\cdots+k^3)+(k+1)^3
\end{aligned}
$$

Therefore, the case that $n = k+1$ is true.
Thus, by mathematical induction, the statement $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$ is true. □

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Proof. By induction, observe that when $n = 1$, then $1^2 = 1^3$ is true. For any $k \geq 1, k \in \mathbb{N}$ assume that $(1+2+3+...+k)^2 = 1^3+2^3+3^3+...+k^3$. Then, to show that $S_k \Rightarrow S_{k+1}$, $(1+2+3+...+k+(k+1))^2 = 1^3+2^3+3^3+...+k^3+(k+1)^3$. Observe that $(1^3+2^3+3^3+...+k^3)+(k+1)^3 = (1+2+3+...+k)^2+(k+1)^3 = (\frac{k(k+1)}{2})^2+(k+1)^3 = \frac{k^2(k+1)^2}{4}+(k+1)^3 = \frac{(k+1)^2}{4}(k^2+4k+4) = \frac{(k+1)^2(k+2)^2}{4} = (\frac{(k+1)(k+2)}{2})^2 = (1+2+3+...+(k+1))^2$. Therefore, by induction, the equation holds for all $n \in \mathbb{N}$. □

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Proof. For $n = 1 : (1)^2 = 1 = 1^3 = 1$ thus it is true for $n = 1$.

Assume it is true for $n = k$ then $(1+2+3+...+k)^2 = 1^3+2^3+3^3+...k^3$.

Now consider $n = k+1$.

Then $(1+2+3+...+k+k+1)^2$.

The left side, $[(1+2+3+...+k)+(k+1)]^2$.

And, $(1+2+3+...+k)^2+2(1+2+3+...+k)(k+1)+(k+1)^2$.

On the right side, $1^3+2^3+3^3+...+k^3+(k+1)[2(1+2+3+...+k)+k+1]$.

Then, $1^3+2^3+3^3+...+k^3+(k+1)[k(k+1)+(k+1)]$.

Then, $1^3+2^3+3^3+...+k^3+(k+1)^2(k+1)$.

Hence, $1^3+2^3+3^3+...+k^3+(k+1)^3$.

Thus it is true for $n = k+1$.

Therefore by mathematical induction $(1+2+3+...+k)^2 = 1^3+2^3+3^3+...+k^3$ for every integer $n \in \mathbb{N}$. □

Comments:

Proposition 2. *For every $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.*

Proof.
Base case: $n = 1(1)^2 = 1^3 = 1$
Base case: $n = 2(1+2)^2 = 1^3+2^3 = 9$
Let $n \in \mathbb{N}$.
Inductive Step: Let $m \in \mathbb{N}$ and suppose $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$ and $(1+2+3+\cdots+m)^2 = 1^3+2^3+3^3+\cdots+m^3$.
We know that $(1+2+3+\cdots+n) = (n^2+n)/2$ due to the proof of Chapter 10 problem 1. This implies that $((n^2+n)/2)^2 = (n(n+1)/2)^2 = 1^3+2^3+3^3+\cdots+n^3$. Let $n = 1$: $((1^2+1)/2)^2 = 1^3 = 1$ so the statement is still true. By the inductive step: $1^3+2^3+3^3+\cdots+m^3 = ((m^2+m)/2)^2$ and $1^3+2^3+3^3+\cdots+m^3+(m+1)^3 = (((m+1)^2+(m+1)/2)^2$ We can deduct that $[1^3+2^3+3^3+\cdots+m^3]+(m+1)^3 = (((m+1)^2+(m+1)/2)^2+(m+1)^3$ where the brackets represent the inductive hypothesis.

$$
\begin{aligned}
&[1^3+2^3+3^3+\cdots+m^3]+(m+1)^3 \\
&= ((m(m+1)^2)/2)^2+(m+1)^3 \\
&= (m^2*(m+1)^2)/4+(m+1)^3 \\
&= (m+1)^2[(m^2)/4+m+1] \\
&= (m+1)^2[(m^2+4m+4)/4] \\
&= ((m+1)^2*(m+2)^2)/2^2 \\
&= ((m+1)(m+2)/2)^2
\end{aligned}
$$

Hence the result is true for n = m+1. Therefore by induction the result is true for all $n \in \mathbb{N}$. ☐

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Proof.

Base Case. Let $n = 1$

Then we have $(1)^2 = 1^3$

Therefore, when $n = 1$, it works.

Inductive Step. Let $k \in \mathbb{N}$. Suppose $(1+2+3+...+k)^2 = 1^3+2^3+3^3+...+k^3$

We know that $(1+2+3+...+k) = \frac{k(k+1)}{2}$

We want to prove that $1^3+2^3+3^3+...+k^3+(k+1)^3 = (\frac{(k+1)(k+2)}{2})^2$

So,

$$\begin{aligned}
1^3+2^3+3^3+...+k^3+(k+1)^3 &= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{(2)^2} + (k+1)^3 \\
&= (k+1)^2(\frac{k^2}{4} + k + 1) \\
&= (k+1)^2(\frac{k^2+4k+4}{4} \\
&= \frac{(k+1)^2(k+2)^2}{2^2} \\
&= (\frac{(k+1)(k+2)}{2})^2
\end{aligned}$$

Therefore, $1^3+2^3+3^3+...+k^3+(k+1)^3 = (\frac{(k+1)(k+2)}{2})^2$

□

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Proof. Let $n \in \mathbb{N}$.

Base case. n = 1. Observe that $1^2 = 1^3 = 1$.

Inductive step. Suppose that $(1+2+3+...+n)^2 = 1^3+2^3+3^3+...+n^3$.

Then $(1+2+3+...+n+(n+1))^2 = [(1+2+3+...+n)+(n+1)]^2$

$= (1+2+3+...+n)^2 + 2(1+2+3+...+n)(n+1)+(n+1)^2$

$= 1^3+2^3+3^3+...+n^3+(n+1)[2(1+2+3+...+n)+(n+1)]$ (inductive hypothesis)

$= 1^3+2^3+3^3+...+n^3+(n+1)[2(\frac{n(n+1)}{2})+(n+1)]$ (by Gauss' formula)

$= 1^3+2^3+3^3+...+n^3+(n+1)[n(n+1)+(n+1)]$

$= 1^3+2^3+3^3+...+n^3+(n+1)(n+1)(n+1)$

$= 1^3+2^3+3^3+...+n^3+(n+1)^3$.

Therefore by induction the proposition holds. □

Comments:

Proposition 2. *For every* $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.

Proof. Base Case: $(n=1): 1^2 = 1^3$

Inductive step: Let $k \in \mathbb{Z}$ and suppose $(1+2+...+k)^2 = 1^3+2^3+...+k^3$

Then, $(1+2+...+k+(k+1))^2 = (1+2+...+k)^2 + 2(k+1)(1+2+...+k)+(k+1)^2$

$= (1^3+2^3+...+^3)+2(k+1)(1+2+...+k)+(k+1)^2$

$= (1^3+2^3+...+k^3)+(k+1)[2(1+2+...+k)+(k+1)]$

$= (1^2+2^3+...+k^3)+(k+1)+(k+1)[2\frac{k(k+1)}{2}+(k+1)]$

$= (1^3+2^3+...+k^3)+(k+1)[k^2+k+k+1]$

$= (1^3+2^3+...+k^3)+(k+1)[(k+1)(k+1)]$

$= (1^3+2^3+...+k^3)+(k+1)^3$

Therefore, $(1+2+...+k+(k+1))^2 = (1^3+2^3+...+k^3)+(k+1)^3$ □

Comments:

Proposition 2. *For every $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.*

Proof. Base Case: $1^2 = 1^3$.

Inductive Step: Let $k \in \mathbb{N}$ and suppose $(1+2+3+...+k)^2 = 1^3+2^3+3^3+...+k^3$.

So, $[(1+2+3+...+k)+(k+1)]^2 = (1+2+3+...+k)^2+2(k+1)(1+2+3+...+k)+(k+1)^2$.

Then, $(1+2+3+...+k)^2+2(k+1)(1+2+3+...+k)+(k+1)^2 = 1^3+2^3+3^3+...+k^3+2(k+1)(1+2+3+...+k)+(k+1)^2$.

Since $(1+2+3+...+k) = (k(k+1))/2$.

We know $1^3+2^3+3^3+...+k^3+2(k+1)(1+2+3+...+k)+(k+1)^2 = 1^3+2^3+3^3+...+k^3+(2k((k+1)^2)/2)+(k+1)^2$.

Then, $1^3+2^3+3^3+...+k^3+(2k((k+1)^2)/2)+(k+1)^2 = 1^3+2^3+3^3+...+k^3+(k+1)((k+1)^2) = 1^3+2^3+3^3+...+k^3+(k+1)^3$

Therefore, $((1+2+3+...+k+(k+1))^2 = 1^3+2^3+3^3+...+k^3+(k+1)^3$

□

Comments:

Proposition 2. *For every $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.*

Proof. Base Case: Suppose $n \in \mathbb{N}$. Let $n = 1$. Then, $(1)^2 = 1 = 1^3$.

Inductive Step: Suppose $k \in \mathbb{N}$ and suppose $(1+2+3+ \quad +k)^2 = 1^3+2^3+3^3+ \quad +k^3$.

We must show that $(1+2+3+ \quad +k+(k+1))^2 = 1^3+2^3+3^3+ \quad +k^3+(k+1)^3$.

Then, $((1+2+3+ \quad +k)+(k+1))^2$

$= (1+2+3+ \quad +k)^2+2(k+1)(1+2+3+ \quad +k)+(k+1)^2$

$= (1^3+2^3+3^3+ \quad +k^3)+2(k+1)(1+2+3+ \quad +k)+(k+1)^2$

$= (1^3+2^3+3^3+ \quad +k^3)+2(k+1)(\frac{k^2+k}{2})+(k+1)^2$ (by ch. 10, #1 proof)

$= (1^3+2^3+3^3+ \quad +k^3)+(k+1)(k^2+k)+(k+1)^2$

$= (1^3+2^3+3^3+ \quad +k^3)+(k+1)k(k+1)+(k+1)^2$

$= (1^3+2^3+3^3+ \quad +k^3)+k(k+1)(k+1)+(k+1)^2$

$= (1^3+2^3+3^3+ \quad +k^3)+k(k+1)^2+(k+1)^2$

$= (1^3+2^3+3^3+ \quad +k^3)+k(k+1)^2+1(k+1)^2$

$= (1^3+2^3+3^3+ \quad +k^3)+(k+1)(k+1)^2$

$= (1^3+2^3+3^3+ \quad +k^3)+(k+1)^3$

Therefore, $(1+2+3+ \quad +k+(k+1))^2 = 1^3+2^3+3^3+ \quad +k^3+(k+1)^3$.

□

Comments:

Proposition 2. *For every $n \in \mathbb{N}$, $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$.*

Proof. The proof is by mathematical induction. Let $n \in \mathbb{N}$.

Base cases:

When $n = 1$, $(1)^2 = 1^3 = 1$

When $n = 2$, $(1+2)^2 = 1^3 + 2^3 = 9$

The base cases are obviously true. Now, suppose that the statement is true when $n = k$ where $k \geq 1$.

Thus, $(1+2+3+\cdots+k)^2 = 1^3+2^3+3^3+\cdots+k^3$

Since $(a+b)^2 = a^2+2ab+b^2$, by expansion, we get $(1+2+3+\cdots+k)^2+2(1+2+3+\cdots+k)(k+1)+(k+1)^2$

Since, $(1+2+3+\cdots+k)^2 = 1^3+2^3+3^3+\ldots n^3$, by substitution, $(1+2+3+\ldots k+(k+1))^2 = (1^3+2^3+3^3+\ldots k^3)+2(1+2+3+\cdots+k)(k+1)+(k+1)^2$

Then, $(1+2+3+\ldots k) = \frac{(k(k+1))}{2}$ By substitution, $(1+2+3+\ldots k+(k+1))^2 = (1^3+2^3+\ldots k^3)+(\frac{(2k(k+1))}{2}+(k+1)^2$

Thus, $(1+2+3+\ldots k+(k+1))^2 = (1^3+2^3+3^3+\ldots k^3)+(k+1)^3$

Comments: