Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. **Base Case** $(n = 12)$**:** Let $n = 12$. n $= 4(3) + 5(0)$, so the proposition is true for $n = 12$.

Inductive step: Let $k \in \mathbb{N}$ and suppose that $k = 4a+5b$ where $a, b \in \mathbb{Z}$ and $a, b \geq 0$. Note that $4(-1)+5(1) = 1$ and that $4(4) + 5(-3) = 1$. Consider $k + 1$. By the inductive hypothesis, $k + 1 = 4a + 5b + 1$. If $a > 0$, we can perform the following substitution:

$$k + 1 = 4a + 5b + (4(-1) + 5(1))$$

Now, $k + 1 = 4(a - 1) + 5(b + 1)$, where $(a - 1)$ and $(b + 1)$ are non-negative integers. If $a = 0$, we can perform a different substitution:

$$k + 1 = 4a + 5b + (4(4) + 5(-3))$$

Now $k + 1 = 4(a + 4) + 5(b - 3)$, where $(a + 4)$ and $(b - 3)$ are non-negative integers. Thus, $k + 1 = 4a + 5b$ for some $a, b \in \mathbb{Z}$ with $a, b \geq 0$.

By induction, the proposition is true. □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. Let $n \in \mathbb{N}$, and suppose that $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$. $12 = 4 \times 3; 13 = 4 \times 2 + 5; 14 = 4 + 5 \times 2 : 15 = 5 \times 3$. For $n \geq 15$, let $k = n \bmod 4 (0 \geq k < 4)$. By induction hypothesis, $n - k = 4a + 5b$ for some $a, b \geq 0$. Thus $n = 4(a + 1) + 5b$, so $n = 4a + 5b$. We proved this through induction. □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. Let $n = 12$. Then, $12 = 4(3) + 5(0)$, where $a = 3$ and $b = 0$. Since a and b are non-negative integers, $n = 12$ is true. Likewise, when $n = 13$, $13 = 4(2) + 5(1)$, where $a = 2$ and $b = 1$.
Let $k \in \mathbb{N}$, such that $k \geq 12$ and $k = 4a + 5b$, where $a, b \geq 0$.
Now, $k + 1 = 4a + 5b + 1 = 4a + 5b + (5 - 4) = 4(a - 4) + 5b + 5 = 4(a - 1) + 5(b + 1)$.
Thus, $k + 1$ is also true. Since k is true and $k + 1$ is true, then there are non-negative integers a and b when $n \geq 12$, such that $n = 4a + 5b$. □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$*

Proof. First, when n=12, we have $12 = 4(3) + 5(0)$, where $a = 3$ and $b = 0$ which verifies the base case because $a, b \in \mathbb{Z}$ and are non-negative. Proving by strong induction, we assume that there is integer $k \geq 12$ such that for every integer $i \in 12, 13, \ldots, k$, there are non-negative integers a, b such that $i = 4a + 5b$. We will use cases to prove that this implies the result for $k + 1$:

case 1: $k = 12$: In this case, $k + 1 = 13 = 4(2) + 5(1)$.

case 2: $k = 13$: For this case, $k + 1 = 14 = 4(1) + 5(2)$, and a pattern is beginning to develop between the coefficients of 4 and 5, decreasing and increasing, respectively.

case 3: $k = 14$: Again, we see that $k + 1 = 4(0) + 5(3)$, reinforcing the pattern.

case 4: $k \geq 15$. Then we can say that $(k + 1) - 4 = k - 3$. This is an integer between 12 and k, and by the inductive hypothesis, there are non-negative integers a, b such that $(k + 1) - 4 = 4a + 5b$. This gives:

$k + 1 = 4a + 5b + 4 = 4(a + 1) + 5b$

Since $(a + 1), b \in \mathbb{Z}$ and are non-negative, this proves the result, thus proving the case and concluding the proof.

□

Comments:

Proposition.
Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.

Proof. Let $n \in \mathbb{N}$.
Base Case 1. $n = 12$. Let $a = 3$ and $b = 0$. Then $n = 4(3) + 5(0)$. So $n = 12$.
Base Case 2. $n = 13$. Let $a = 2$ and $b = 1$. Then $n = 4(2) + 5(1)$. So $n = 13$.
Base Case 3. $n = 14$. Let $a = 1$ and $b = 2$. Then $n = 4(1) + 5(2)$. So $n = 14$.
Base Case 4. $n = 15$. Let $a = 0$ and $b = 3$. Then $n = 4(0) + 5(3)$. So $n = 15$.

Prove $n = 4a + 5b$ for all non-negative integers a and b where $n \geq 12$.

Strong Inductive Hypothesis.
Suppose that for some $n \geq 12$, we also have that $12 \leq k \leq n$, there exists non-negative integers a and b such that $k = 4a + 5b$.

Inductive Step. Show that $n + 1 \geq 16$.

Realize that $n + 1$ is the same as $n - 3 + 4$. Where $n - 3 \geq 12$.

Using strong inductive hypothesis, $n - 3 = 4a + 5b$ for non-negative integers a and b.

Notice that $n + 1$ can be written as $4(a + 1) + 5b$.

Therefore, by strong induction, there are non-negative integers a and b such that $n = 4a + 5b$ where $n \in \mathbb{N}$ and $n \geq 12$.

□

Comments:

Proposition 1. *Let $n \in \mathbb{Z}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$*

Proof. Let $n, a, b \in \mathbb{Z}$ and suppose $n \geq 12$ and $a, b \geq 0$
Let $n = 12$, $a = 3$, $b = 0$.
Then,

$$12 = 4(3) + 5(0)$$
$$12 = 12 + 0$$
$$12 = 12$$

For the sake of contradiction, suppose $n \neq 4a + 5b$ for all $n \geq 12$ and $a, b \geq 0$.
Let $k > 12$ be the smallest integer for which $k \neq 4a + 5b$.
For $k - 1 = 4a + 5b$,

$$k = 4a + 5b + 1$$
$$k = 4a + 5b + 5 - 4$$
$$k = 4(a - 1) + 5(b + 1)$$

This means that $k = 4a + 5b$ since $k = 4(a - 1) + 5(b + 1)$ contradicts the form $k \neq 4a + 5b$.
Thus, it was wrong to assume that it is untrue that $n = 4a + 5b$ for all $n \geq 12$.
Therefore, $n = 4a + 5b$ for all $n \geq 12$. □

Comments:

Proposition 1. *Let $n \in \mathbb{Z}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$*

Proof. Base Case:

$$n = 12 \Rightarrow 4(3) + 5(0)$$
$$n = 13 \Rightarrow 4(2) + 5(1)$$
$$n = 14 \Rightarrow 4(1) + 5(2)$$
$$n = 15 \Rightarrow 4(0) + 5(3)$$

I.H. Suppose $n \in \mathbb{N}$ and $12 \leq n \leq k$ and $k \geq 15, k \in \mathbb{N}$. Now we must prove that $S_k \Rightarrow S_{k+1}$

If $k \geq 15$, then $k+1 = k-3+4$ where $k-3 \geq 12$. We then observe that $k-3 = 4a+5b$ for some non-negative integers a, b. Hence, because you add another 4, $k+1 = k-3 = 4(a+1)+5b$, where $a+1$ and b are non-negative integers. Thus, by strong induction, the equation holds. □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. Let $n \in \mathbb{N}$ and suppose $n \geq 12$.

Base Case: Let $n = 12$, then $12 = 4a + 5b$ which holds for $a = 3$ and $b = 0$,
$12 = 4(3) + 5(0)$.
Since $n \geq 12$, let $n = 13$, then $a = 2$ and $b = 1$, $13 = 4(2) + 5(1)$.
Also $n = 14$, which holds for $a = 1$ and $b = 2$, $14 = 4(1) + 5(2)$.
And $n = 15$, which holds for $a = 0$ and $b = 3$, $15 = 4(0) + 5(3)$.
Then $n = 16$, which holds for $a = 4$ and $b = 0$, $16 = 4(4) + 5(0)$.
Inductive Step: Let $k \in \mathbb{N}, k \geq 12, k = 4a + 5b$.
Consider $n = (k+1) - 5$, since $k \geq 16$ was proved by the base cases, then $m \geq 12$.
So $k - 4$ can be plugged in $k - 4 = 4a + 5b$.
Thus $k = 4(a+1) + 5b$, and $a, b \in \mathbb{N}$.
Therefore the proposition is true by induction. □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. Proof by strong induction. Let $n \in \mathbb{N}$. The base cases are given below.

$$n = 12 : 4(3) + 5(0) = 12$$
$$n = 13 : 4(2) + 5(1) = 13$$
$$n = 14 : 4(1) + 5(2) = 14$$
$$n = 15 : 4(0) + 5(3) = 15$$

The inductive hypothesis is given as follows, if $n \in \mathbb{N}$ and $n > 15$. Assume if $k \in \mathbb{N}$ with $12 \leq k < n$, then there are non-negative integers a, b with $4a + 5b = k$. Consider $n = (n-4) + 4$. Then $n - 4 > 15 - 4 = 11$, so $12 \leq n - 4 < n$. By the inductive hypothesis $\exists a, b \in \mathbb{Z}$, where $a, b \geq 0$, and $4a + 5b = n - 4$. If $4a + 5b = n - 4$ then

$$n - 4 - 4a = 5b$$
$$n - 4(1 + a) = 5b$$
$$n = 5b + 4(a + 1)$$

Hence by strong induction the proof is true that if $n \geq 12$ then there are non-negative integers a and b such that $n = 4a + 5b$. □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof.

Base case 1. Suppose $n \geq 12$ and $n = 4a + 5b$ with a and b being non-negative integers.
$n = 12 = 4(3) + 5(0)$
$n = 13 = 4(2) + 5(1)$
$n = 14 = 4(1) + 5(2)$
$n = 15 = 4(0) + 5(3)$

Induction 1. Suppose $12 \geq m \geq k$ where $m = 4a + 5b$.
Hence, by strong induction hypothesis, we have that $n - 2 = 4a + 5b$ for some non-negative integers a and b.
Then, $n + 1 = n - 2 + 3 = 4(a + 1) + b$ where a+1 and b are non-negative integers
Therefore, by strong induction, there are non-negative integers a and b such that $n = 4a + 5b$.

□

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. Let $n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $n \geq 12$, and $a, b \geq 0$.

Let n = 12. Then $12 = 4a + 5b$. Observe that when $a = 3$ and $b = 0$, it follows that $4(3) + 5(0) = 12$.

For sake of contradiction, suppose $n \neq 4a + 5b$ for all $a, b \geq 0$. Let $k > 12$ be the smallest integer for which $k \neq 4a+5b$. Then $k-1 = 4a+5b$, so $k = 4a+5b+1$. It follows that $k = 4a+5b+1 = 4a+5b+5-4 = 4(a-1)+5(b+1)$, which is a contradiction since $(a-1), (b+1) \in \mathbb{Z}$ and $a-1 \geq 0$ since $a = 3$ in the smallest case of $n = 12$. Therefore $n = 4a + 5b$ is true for every $n \geq 12$. □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. by induction:

Base Case:

$$\begin{aligned}(n = 12) &: 12 = 4a + 5b = 4(3) + 5(0) = 12\\ (n = 13) &: 13 = 4a + 5b = 4(2) + 5(1) = 13\\ (n = 14) &: 14 = 4a + 5b = 4(1) + 5(2) = 14\\ (n = 15) &: 15 = 4a + 5b = 4(0) + 5(3) = 15\end{aligned} \tag{1}$$

Inductive Step (strong induction): Suppose for some $n, k \in \mathbb{N} \geq 8$, For every $8 \leq n \leq k$, there exist non-negative integers such that $k = 4a + 5b$. Suppose $k \geq 16$. By Assumption, k-3 is true. By the Inductive Hypothesis, $k - 3 = 4a + 5b$ for some $a, b \in \mathbb{N}$ because $k - 3 \geq 12$.Therefore, we know k+1 is true because $(k - 3) + 4 = k + 1$. □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. Suppose $n \in \mathbb{N}$ and $n \geq 12$.

Suppose $a, b \in \mathbb{Z}$ and $a, b \geq 0$.

Base Cases: Let $n = 12$. Then, $12 = 4(3) + 5(0)$.
Let $n = 13$. Then, $13 = 4(2) + 5(1)$.
Let $n = 14$. Then, $14 = 4(1) + 5(2)$.
Let $n = 15$. Then, $15 = 4(0) + 5(3)$.
Let $n = 16$. Then, $16 = 4(4) + 5(0)$.

Inductive Step: Let $k \in \mathbb{N}$ and $k \geq 12$. Suppose $\forall l \in \mathbb{N}$ with $12 \leq l < k$, $l = 4a + 5b$.
Let $l = k - 5$.
Moreover, $l + 1 = 4a + 5b$
$(k - 5) + 1 = 4a + 5b$ (by Inductive Hypothesis)
$k - 4 = 4a + 5b$
$k = 4a + 5b + 4$
$k = (4a + 4) + 5b$
$k = 4(a + 1) + 5b$ and $a + 1 \in \mathbb{N}$
Thus, k matches the form of n.
Therefore, $k + 1 = 4a + 5b$.

□

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. Base Case: Let $n = 12$. Then consider $a = 3$ and $b = 0$. Thus both a and b are non-negative integers and $12 = 4a + 5b = 4 * 3 + 5 * 0 = 12 + 0 = 12$.

Inductive Step: Let $k \in \mathbb{N}$ and $k \geq 12$. If $k = 4a+5b$ for some non-negative integers a and b, then $k+1 = 4c+5d$ for some non-negative integers c and d.

Consider $c = a-1$ and $d = b+1$. Then $k+1 = 4c+5d = 4(a-1)+5(b+1) = 4a-4+5b+5 = 4a+5b+1 = k+1$.

Therefore, for $n \geq 12$ and $n \in \mathbb{N}$, there are non-negative integers a and b such that $n = 4a + 5b$ □

Comments:

Proposition 1. *Let $n \in \mathbb{N}$. If $n \geq 12$, then there are non-negative integers a and b such that $n = 4a + 5b$.*

Proof. The proof is by strong induction. Let $n \in \mathbb{N}$ and let $a, b \in \mathbb{Z} \geq 0$.

Base case. $n = 12$ can be written as $4(3) + 5(0)$ where $a = 3$ and $b = 0$.

Strong inductive hypothesis. Suppose that for some natural number $n \geq 12$, there exists an integer k where $12 \leq k \leq n$ such that $k = 4a + 5b$.

Inductive step.

Observe $n = 13$. Then, $n + 1 = 14$. $14 = 4(1) + 5(2)$ where $a = 1$ and $b = 2$.

Observe $n \geq 14$. Then, $n+1 = n-2+3$ where $n-2 \geq 12$. Thus, $n-2 = 4a+5b$ for some non-negative integers a and b. Since $n - 2 + 3 = 3(a + 1) + b$ where a+1 and b are non-negative integers, then there are non-negative integers a and b such that $n = 4a + 5b$.

Comments: