CONTRAPOSITIVE PROOFS (AND CONGRUENCE)

1. Let $a, b \in \mathbb{Z}$. Prove that if ab and $a+b$ are both even, then both a and b are even. (Use the contrapositive, but be careful to get the contrapositive right).

2. Let $a \in \mathbb{N}$.

a) Prove that if $2^a - 1$ is prime, then a is odd or $a = 2$.

b) Is the converse true?

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Definition. Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$.

i) We say that a and b are **congruent modulo** n if $n|(a-b)$. This is expressed in symbols as $a \equiv b \pmod{n}$.

ii) The **congruence class of** a **modulo** n is $[a]_n = \{x \in \mathbb{Z} : x \equiv a \pmod{n}\} = \{x \in \mathbb{Z} : n|(a-x)\}$.

Note that $n|(a-b)$ is just a short way of saying that there is $k \in \mathbb{Z}$ such that $nk = a - b$ or, equivalently, $a = nk + b$. This gets us back near the division algorithm, which says that there are unique integers q and r such that $a = nq + r$ and $0 \leq r < n$. It may help to think of this when working on the following problems.

3. By definition, an integer a is congruent to 0 modulo 2 if $2|(a-0)$, in other words, if a is even. Thus $[0]_2 = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$.

a) What is the congruence class of 4 modulo 2? Is this a new set?

b) Write out the congruence class of 1 modulo 2.

c) Have we found all the congruence classes modulo 2?

d) Identify the congruence classes of 0, 1, and 2 modulo 3. Are there any other congruence classes modulo 3?

e) How many congruence classes do you expect to find modulo 4?

4. The congruence classes modulo 10 are $[0]_{10}$, $[1]_{10}$, $[2]_{10}$, $[3]_{10}$, $[4]_{10}$, $[5]_{10}$, $[6]_{10}$, $[7]_{10}$, $[8]_{10}$, and $[9]_{10}$.

a) Write out some of the congruence classes modulo 10. What do the numbers in a given congruence class have in common?

b) Let $n \in \mathbb{N}$. What are the possible congruence classes of 3^n modulo 10?

c) What is the last digit of 3^{2021}?

Challenge. Let $a, b \in \mathbb{Z}$. Prove that $(a+b)^3 \equiv a^3 + b^3 \pmod{3}$.