Determine if each statement is true or false. If it is true, prove it. If it is false, give a disproof.

1. There is an integer \( n \) such that \( n^2 \equiv -1 \pmod{13} \).

**Solution.** The statement is true. An example suffices to prove that such an integer exists: \( 2(13) = 26 = 25 \equiv (-1) \). Hence \( 5^2 \equiv -1 \pmod{13} \).

2. For all \( a, b, c \in \mathbb{Z} \), if \( a \mid bc \), then \( a \mid b \) or \( a \mid c \).

**Solution.** The statement is false. A counterexample suffices to prove the negation: there are \( a, b, c \in \mathbb{Z} \) such that \( a \mid bc \) but \( a \nmid b \) and \( a \nmid c \). Let \( a = 6, b = 2, \) and \( c = 3 \). We see that \( 6 \mid 6 \) (so \( a \mid bc \)) but \( 6 \nmid 2 \) (so \( a \nmid b \)) and \( 6 \nmid 3 \) (so \( a \nmid c \)).

3. There are integers \( a \) and \( b \) such that \( 12a + 15b = 2 \).

**Solution.** The statement is false. A disproof entails proving that for any integers \( a \) and \( b \) it follows that \( 12a + 15b \neq 2 \). We proceed by contradiction. Suppose \( a \) and \( b \) are integers such that \( 12a + 15b = 2 \). It follows that \( 3(4a + 5b) = 2 \) and thus that \( 3 \mid 2 \). This is a contradiction.

4. There are integers \( a \) and \( b \) such that \( 11a + 15b = 1 \).

**Solution.** The statement is true. An example suffices to prove it: \( 11(-4) + 15(3) = 1 \).

5. If \( X \subseteq A \cup B \), then \( X \subseteq A \) or \( X \subseteq B \).

**Solution.** The statement is false. A counterexample suffices to prove the negation: there are sets \( X, A, \) and \( B \) such that \( X \subseteq A \cup B \) but \( X \nsubseteq A \) and \( X \nsubseteq B \). Let \( X = \{1, 2\}, A = \{1\}, \) and \( B = \{2\} \). Then \( A \cup B = \{1, 2\} \supseteq X \) but \( X \nsubseteq A \) and \( X \nsubseteq B \).

**Challenge.** There is an integer \( n \) such that \( n^2 \equiv -1 \pmod{3} \).

**Solution.** This statement is false. A disproof entails proving that for any integer \( n \) it follows that \( n^2 \neq -1 \pmod{3} \). The proof I have in mind is direct and involves three cases: \( n \equiv 0 \pmod{3}, n \equiv 1 \pmod{3}, \) and \( n \equiv 2 \pmod{3} \). We can then determine that in the first case \( n^2 \equiv 0 \pmod{3} \) and in the latter two cases \( n^2 \equiv 1 \pmod{3} \). In none of these cases is \( n^2 \equiv -1 \pmod{3} \).