THE CANTOR SET

Definition. We construct the Cantor middle-thirds set as follows. Define the following sets (as unions of intervals in $\mathbb{R}$):

$$\begin{aligned} C_0 &= [0,1] \\ C_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\ C_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\ &\vdots \end{aligned}$$

In general, C_{k+1} is formed by removing the open middle third of each interval in C_k. The **Cantor middle-thirds set** is

$$\boxed{\mathcal{C} = \bigcap_{k=0}^{\infty} C_k}$$

1. Prove or disprove the following statements about $\mathcal{C}$.

a) $\mathcal{C} = \emptyset$

b) $\forall n \in \mathbb{N}, \left(0, \frac{1}{2^n}\right) \not\subseteq \mathcal{C}$

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Definition. Any real number can be expressed in ternary, which is like binary but with 3 possible values for each digit. For example, the ternary number 201 is

$$\left(2 \times 3^2\right) + \left(0 \times 3^1\right) + \left(1 \times 3^0\right) = 18 + 0 + 1 = 19.$$

And the ternary number 0.201 is

$$\left(2 \times 3^{-1}\right) + \left(0 \times 3^{-2}\right) + \left(1 \times 3^{-3}\right) = \frac{2}{3} + \frac{0}{9} + \frac{1}{27} = \frac{19}{27}.$$

Proposition 1. *Let $x \in [0,1]$. Then $x \in \mathcal{C}$ if and only if x can be written as a ternary number using* ***only*** *the digits 0 and 2.*

2. Is 1/4 in the Cantor set?

3. Is the Cantor set countable?

Challenge. Prove proposition 1.