One percent of the population are carriers for a disease. A test for the disease is positive 90% of the time when the person tested is a carrier. The test is also positive 5% of the time when the person is not a carrier.

Let T be the event that the test is positive. Let C be the event that the person is a carrier. We can summarize the given information:

$$\mathrm{P}(C) = 0.01 \quad \mathrm{P}(T|C) = 0.9 \quad \mathrm{P}(T|C') = 0.05.$$

a) What is the probability that two tests run independently on blood samples from the same person will return the same result?

Let T_1 be the event that the first test is positive and T_2 be the event that the second test is positive. The tests are run independently so you might think that $\mathrm{P}(T_1 \cap T_2) = \mathrm{P}(T_1)\,\mathrm{P}(T_2)$. This is *not true* because the two samples come from the same person. It still matters if the person is a carrier or not. Independent (but identical) tests means that

$$\begin{aligned}\mathrm{P}(T_1 \cap T_2|C) &= \mathrm{P}(T_1|C)\,\mathrm{P}(T_2|C)\\ &= \mathrm{P}(T|C)^2\\ &= 0.81\end{aligned}$$

and

$$\begin{aligned}\mathrm{P}(T_1 \cap T_2|C') &= \mathrm{P}(T_1|C')\,\mathrm{P}(T_2|C')\\ &= \mathrm{P}(T|C')^2\\ &= 0.0025.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{P}(T_1 \cap T_2) &= \mathrm{P}(T_1 \cap T_2|C)\,\mathrm{P}(C) + \mathrm{P}(T_1 \cap T_1|C')\,\mathrm{P}(C') \quad \text{(by the law of total probability)}\\ &= 0.81(0.01) + 0.0025(0.99)\\ &= 0.010575.\end{aligned}$$

We can similarly calculate that

$$\begin{aligned}\mathrm{P}(T_1' \cap T_2') &= \mathrm{P}(T_1' \cap T_2'|C)\,\mathrm{P}(C) + \mathrm{P}(T_1' \cap T_1'|C')\,\mathrm{P}(C')\\ &= \mathrm{P}(T_1'|C)\,\mathrm{P}(T_2'|C)\,\mathrm{P}(C) + \mathrm{P}(T_1'|C')\,\mathrm{P}(T_2'|C')\,\mathrm{P}(C')\\ &= \mathrm{P}(T'|C)^2\,\mathrm{P}(C) + \mathrm{P}(T'|C')^2\,\mathrm{P}(C')\\ &= (0.1)^2(0.01) + (0.95)^2(0.99)\\ &= 0.893575.\end{aligned}$$

The probability that both tests return the same result is thus

$$\mathrm{P}(T_1 \cap T_2) + \mathrm{P}(T_1' \cap T_2') = 0.010575 + 0.893575 = 0.90415.$$

b) What is the probability that a person has the disease if both tests are positive?

Now we are looking for $\mathrm{P}(C|T_1 \cap T_2)$. Bayes' rule connects this to $\mathrm{P}(T_1 \cap T_2|C)$, which we have already calculated. We have

$$\begin{aligned}\mathrm{P}(C|T_1 \cap T_2) &= \frac{\mathrm{P}(T_1 \cap T_2|C)\,\mathrm{P}(C)}{\mathrm{P}(T_2 \cap T_2)} \quad \text{(Bayes' rule)}\\ &= \frac{(0.81)(.01)}{0.010575} \quad \text{(using calculations above)}\\ &\approx 0.76596.\end{aligned}$$