A *continuous random variable* is a random variable which takes a continuum of possible values. The cumulative distribution function (cdf) is the same as ever ($F(x) = P(X \le x)$ for any random variable X).

Definition. Let X be a continuous random variable. A **probability density function** (pdf) for X is any function f such that

$$P(a < X < b) = \int_a^b f(x)dx$$

for any real numbers a and b with $a < b$.

Note that the probability density function of a continuous random variable does not give probabilities directly: the values of $f(x)$ are not probabilities. In contrast, the probability distribution function of a discrete random variable gives probabilities directly. Making the distinction between discrete and continuous random variables is essential.

Thoerem. *A function f may be the probability density function of a continuous random variable if and only if*

1. $f(x) \ge 0$ *for all x and*
2. $\int_{-\infty}^{\infty} f(x)dx = 1.$

Thoerem. *If X is a continuous random variable, then*

1. $P(X = a) = 0$ *for every number a*
2. $P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$ *for every numbers a and b*

Definition. Let X be a continuous random variable and let $f(x)$ be a probability distribution function for X. The *expected value* (or *mean*) of X is $\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$. The *median* is the number $\tilde{\mu}$ such that $0.5 = \int_{-\infty}^{\tilde{\mu}} f(x)dx$. In terms of the cdf F, $\tilde{\mu}$ is the number such that $F(\tilde{\mu}) = 0.5$ (equivalently, $\tilde{\mu} = F^{-1}(0.5)$).

The usual calculating formula for the variance still works: $\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$. For a continuous random variable $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$.

Definition. Let X be any random variable (continuous or discrete). The cumulative distribution function of X is $F(x) = P(X \le x)$.

Thoerem. *If X is a continuous random variable with probability density function f and cumulative distribution function F, then*

1. $F(x) = \int_{-\infty}^{x} f(t)dt$
2. $f(x) = F'(x)$.

Thoerem. *If X is a discrete random variable with probability distribution function f and cumulative distribution function F, then*

1. $F(x) = \sum_{n \le x} f(n)$ *(where the sum is over all possible values of X that are less than or equal to x)*
2. $f(x) = \left[\lim_{t \to x^+} F(t)\right] - \left[\lim_{t \to x^-} F(t)\right]$.

Special continuous distributions:

A random variable has a *uniform continuous distribution* on the interval (a, b) if its probability density function is

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

In this case its expected value is $\mu = \dfrac{a+b}{2}$ and its variance is $\sigma^2 = \dfrac{(b-a)^2}{12}$.

A random variable has an *exponential distribution* with parameter $\lambda > 0$ if its probability density function is

$$g(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

In this case its expected value is $\mu = \dfrac{1}{\lambda}$ and its variance is $\sigma^2 = \dfrac{1}{\lambda^2}$.

A random variable has a *normal distribution* with parameters μ and $\sigma > 0$ if its probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

In this case its expected value is μ and its variance is σ^2. If $\mu = 0$ and $\sigma = 1$, then the random variable has a *standard normal distribution.* The cumulative distribution function of a standard normal random variable is $\Phi(x)$ and its values are given in table A.3 of the textbook.

Special discrete distributions:

A random variable has a *Poisson distribution* with parameter $\lambda > 0$ if its probability distribution function is

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ for } x = 0, 1, 2, \ldots$$

In this case its expected value is $\mu = \lambda$ and its variance is $\sigma^2 = \lambda$.

A random variable has a *binomial distribution* with parameters n and θ (with n a positive integer and $0 < \theta < 1$) if its probability distribution function is

$$b(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \text{ for } x = 0, 1, 2, \ldots, n$$

In this case its expected value is $\mu = n\theta$ and its variance is $\sigma^2 = n\theta(1-\theta)$.

Thoerem. *If X is a binomial random variable with parameters n and θ and both $n\theta \geq 10$ and $n(1-\theta) \geq 10$, then X is approximately normal with mean $\mu = n\theta$ and $\sigma = \sqrt{n\theta(1-\theta)}$.*

Example 1. If X is binomial with parameters $n = 100$ and $\theta = 0.3$, then

$$P(X \leq 25) \approx P\left(Z \leq \frac{25.5 - 100(0.3)}{\sqrt{100(0.3)(1-0.3)}}\right) \approx \Phi(-0.98) = 0.1635$$

Here we have used the correction for continuity, which is why we have 25.5 instead of 25.

Example 2. Find the mean of the random variable with cdf $F(x) = \begin{cases} 1 - \frac{1}{x^2} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$. We first must find the pdf $f(x) = F'(x) = \begin{cases} \frac{2}{x^3} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$. Then $\mu = \displaystyle\int_{-\infty}^{\infty} x f(x) dx = \int_1^{\infty} \frac{2}{x^2} dx = \left[-\frac{2}{x}\right]_1^{\infty} = 2$. Note that the variance of this random variable is undefined because $\displaystyle\int_{-\infty}^{\infty} x^2 f(x) dx$ is a divergent improper integral.