Note. A standard deck of cards has 52 cards in 4 suits: spades ♠, hearts ♡, clubs ♣, and diamonds ♢. Spades and clubs are black suits, while hearts and diamonds are red suits. Within each suit there are 13 ranks: 2, 3, 4, ..., 10, Jack, Queen, King, and Ace. Together, a rank and a suit uniquely identify the card (cards are 2-dimensional).

1. Suppose 10 cards are dealt from a well-shuffled deck.

a) Find the probabilty that exactly 5 are hearts.

$$\frac{\binom{13}{5}\binom{39}{5}}{\binom{52}{10}} = \text{dhyper}(5, 13, 39, 10) \approx 0.046839$$

b) Find the probability that 5 or more are hearts.

$$\sum_{x=5}^{10} m(x) = 1 - \text{phyper}(4, 13, 39, 10) \approx 0.056814$$

2. Suppose now that you shuffle the deck and look a the top card 10 times.

a) Find the probabilty that exactly 5 of the cards you see are hearts.

$$\binom{10}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^5 = \text{dbinom}(5, 10, 0.25) \approx 0.058399$$

b) Find the probability that 5 or more are hearts.

$$\sum_{x=5}^{10} m(x) = 1 - \text{pbinom}(4, 10, 0.25) \approx 0.0781269|$$

3. For this problem, suppose you have created a mega deck by shuffling together 10 regular decks (for a total of 520 cards).

a) Find the probability that exactly 5 are hearts if you deal 10 cards from the mega deck.

$$\frac{\binom{130}{5}\binom{390}{5}}{\binom{520}{10}} = \text{dhyper}(5, 130, 390, 10) \approx 0.057444$$

b) Find the probability that exactly 5 cards are hearts when you shuffle and look at the top card 10 times.

$$\binom{10}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^5 = \text{dbinom}(5, 10, 0.25) \approx 0.058399 \quad \text{(same as 2a)}$$

c) Repeat both parts a and b for a super-mega deck of 5200 cards.

a) $$\frac{\binom{1300}{5}\binom{3900}{5}}{\binom{5200}{10}} = \text{dhyper}(5, 1300, 3900, 10) \approx 0.058305$$

b) same as before

d) Any observations?

When taking a small sample from a large pop, with and without replacement are about the same. $\frac{n}{N} \leq 0.05$ rule of thumb.

4. Return to a regular deck of 52 cards. The goal this time is to find the PMF of a new distribution in which you count the number of cards you must look at to find a heart. For this problem, take the inefficient approach of shuffling and looking at just the top card. Return the card to the deck an repeat until you see a heart. Let X be the number of times you do this (including the time when you see the heart and stop).

a) What are the possible values for X?

$1, 2, 3, 4, \ldots$

b) Find the PMF for X.

x	1	2	3	4	...
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{4}\left(\frac{1}{4}\right)$	$\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$	$\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)$	...

$$P(X=x) = \left(\frac{3}{4}\right)^{x-1}\left(\frac{1}{4}\right) \quad \text{for } x = 1, 2, 3, \ldots$$

5. This is the more reasonable version of the previous problem: deal cards from a well-shuffled deck until you deal a heart. Let Y be the number of cards you deal (including the heart).

a) What are the possible values for Y?

$1, 2, 3, 4, \ldots, 40.$

b) Find the PMF for Y.

y	1	2	3	...
$P(Y=y)$	$\frac{13}{52}$	$\frac{39}{52}\left(\frac{13}{51}\right)$	$\frac{39}{52}\left(\frac{38}{51}\right)\left(\frac{13}{50}\right)$	...

$$P(Y=y) = \frac{39!}{(40-y)!}\left[\frac{(52-y)!}{52!}\right] 13$$

6. Suppose you're studying abroad in Florence and, as you leave for school one morning, you find a coin in the pocket of your jacket. You don't look at the coin, but you can tell it's either a €1 or a €2 coin; you figure both are equally likely. On your way out, a friend pays you back for the coffee you bought the other day with a €1 coin that you drop in the same pocket. Then you stop for coffee and randomly select a coin from the pocket: it's a €1 coin. What is the probability that the remaining coin is €2?

Events: A: original coin is €1; B: select €1.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{1\left(\frac{1}{2}\right)}{1\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)} = \frac{1/2}{3/4} = \frac{2}{3}.$$

so $P(A^c|B) = \frac{1}{3}$