FUNCTIONS OF RANDOM VARIABLES II

Theorem 1. Let X be a random variable and let a and b be constants.

i) $E(aX+b) = aE(X)+b$

ii) $\text{Var}(aX+b) = a^2\text{Var}(X)$

Theorem 2. For any random variables $X_1, X_2, \ldots, X_n$:

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) \text{ and } \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i=1}^{j}\sum_{j=1}^{n} \text{Cov}(X_i, X_j)$$

Definition. Let X and Y be random variables with means μ_X and μ_Y and standard deviations σ_X and σ_Y, respectively. The **covariance** of X and Y is

$$\text{Cov}(X,Y) = E\left[(X-\mu_x)(Y-\mu_Y)\right] = E(XY) - \mu_X\mu_Y$$

The **correlation coefficient** of X and Y is

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}$$

The correlation coefficient is a unit-less measure that is always between -1 and 1. It measures the strength and nature of the relationship between X and Y.

Definition. Let X and Y be jointly-distributed random variables with joint PMF/PDF $f(x,y)$. Then X and Y are **independent** if $f(x,y) = f_X(x)f_Y(y)$ for all x and y.

Theorem 3. If $X_1, X_2, \ldots, X_n$ are **independent** random variables, then $\text{Cov}(X_i, X_j) = 0$ for all $i \neq j$ and

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i)$$

Theorem 4. If X and Y are independent, normally-distributed random variables and a and b are constants, then $aX+bY$ is normally-distributed.

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1. The width of a casing for a door in normally-distributed with a mean of 24 inches and a standard deviation of 0.125 inches. The width of a door is normally-distributed with a mean of 23.875 inches and a standard deviation of 0.0625 inches. Assume the door and the casing widths are independent of each other.

a) Determine the mean and standard deviation of the difference between the width of the casing and the width of the door.
b) What is the probability that the the difference in widths (casing minus door) is bigger than 0.25 inches?
c) What is the probability that the door doesn't fit in the casing?

2. A manufacturer produces roller bearings, each of which is a cylinder with a length and radius that vary slightly. The bearings are made from a material with a density of 8 g/cm^3.

a) Express the mass of a bearing as a function of the length and radius.

b) Suppose the radius is constant at 0.2 cm and the length is random with a mean of 1.25 cm and a standard deviation of 0.05 cm. Find the expected value and standard deviation of the mass.

c) Suppose the length is constant at 1.25 cm and the radius is random with a mean of 0.2 cm and a standard deviation of 0.01 cm. Find the expected value of the mass.

d) Explain why you can't find the standard deviation of the mass when the radius varies.

Theorem 5 (Propagation of error formula)**.** If X is a random variable with mean μ_X and standard deviation σ_X and $g(x)$ is a differentiable function, then

i) $E\left[g(X)\right] \approx g(\mu_X)$

ii) $\text{Var}\left[g(X)\right] \approx \left[g'(\mu_X)\sigma_X\right]^2$

3. Use the propagation of error formula to estimate the standard deviation of the mass of the bearing in part c of the last problem.

Theorem 6 (Propagation of error formula II)**.** If $X_1, X_2, \ldots, X_n$ are random variables with mean $\mu_1, \mu_2, \ldots, \mu_n$ and standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_n$, respectively and $g(x_1, x_2, \ldots, x_n)$ is a differentiable function, then

i) $E\left[g(X_1, X_2, \ldots, X_n)\right] \approx g(\mu_1, \mu_2, \ldots, \mu_n)$

ii) $\text{Var}\left[g(X_1, X_2, \ldots, X_n)\right] \approx \sum_{i=1}^{n} \left[\left(\frac{\partial g}{\partial x_i}\right)\sigma_i\right]^2$ (where the partial derivatives are evaluated at $(\mu_1, \mu_2, \ldots, \mu_n)$.

4. Find the expected value and variance of the mass of a bearing when both the length and radius are random:

- the length is random with a mean of 1.25 cm and a standard deviation of 0.05 cm
- the radius is random with a mean of 0.2 cm and a standard deviation of 0.01 cm.

5. The goal of this problem is to find the correlation coefficient for the two random variables in the die-coin experiment from worksheet 2. I think it will reduce confusion if we re-label the two random variables as X (the number of heads flipped) and Y (the roll of the die). The joint PMF is given below.

$f(x,y)$		$x = 0$	1	2	3	4	$f_Y(y)$
y	1	$\frac{1}{8}$	$\frac{1}{8}$	0	0	0	
	2	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	0	0	
	3	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{1}{32}$	0	
	4	$\frac{1}{64}$	$\frac{4}{64}$	$\frac{6}{64}$	$\frac{4}{64}$	$\frac{1}{64}$	
$f_X(x)$							

a) Add rows to find the marginal PMF for Y. Make sure this gives you what you expected.
b) Add columns to find the marginal PMF for X. Make sure the sum of the values is 1.
c) Use the marginal PMFs to find $E(X)$ and $E(Y)$.
d) Calculate $\text{Var}(X)$ and $\text{Var}(Y)$.
e) Calculate $E(XY)$.
f) Calculate $\text{Cov}(X,Y)$.
g) Find $\rho_{X,Y}$.