BOUNDS AND BIJECTIONS

Lemma (Triangle inequality). *For all $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$.*

Definition. A function $f : D \to \mathbb{R}$ is **bounded** if there is $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$.

1. Use the triangle inequality to show that the function $f : [-3, 5] \to \mathbb{R}$ defined by $f(x) = x^2 - 6x + 1$ is bounded.

Definition. Let $f : D \to \mathbb{R}$ be a function. Then $f(D) := \{f(x) : x \in D\}$ and

$$\sup_{x \in D} f(x) := \sup f(D) \text{ and } \inf_{x \in D} f(x) := \inf f(D).$$

2. Find $\inf_{x \in D} f(x)$ and $\sup_{x \in D} f(x)$ for the function in the previous problem.

3. Does your answer to the previous problem change if we change the domain to $(-3, 5)$ but otherwise keep the function the same?

Date: Due January 31, 2022.

Proposition. *Let D be a nonempty set and $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be bounded functions such that $\forall x \in D,\ f(x) \leq g(x)$. Then*

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \text{ and } \inf_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

4. Prove that it is possible for f and g to satisfy the conditions of the previous proposition but still have $\inf_{x \in D} g(x) \leq \sup_{x \in D} f(x)$. (This means producing an example of two bounded functions f and g such that $\forall x \in D,\ f(x) \leq g(x)$ and $\inf_{x \in D} g(x) \leq \sup_{x \in D} f(x)$).

Definition. Sets A and B have the same cardinality (notation: $|A| = |B|$) if there is a bijection $f : A \to B$.

5. Prove that the real intervals $(0, 1)$ and $(0, \infty)$ have the same cardinality by producing a bijection.

Challenge. Prove that $(0, 1)$ and $\mathbb{R}$ have the same cardinality by producing a bijection. (The composition of two bijections is again a bijection so it might be helpful to work in two steps: e.g. $(0, 1) \to (-1, 1) \to \mathbb{R}$)