MORE SEQUENCES (YAY!)

Definition. Let $\{x_n\}$ be a bounded sequence. Define the following:

$$\begin{aligned} a_n &= \sup\{x_k : k \geq n\} \\ b_n &= \inf\{x_k : k \geq n\} \\ \limsup_{n\to\infty} x_n &= \lim_{n\to\infty} a_n \\ \liminf_{n\to\infty} x_n &= \lim_{n\to\infty} b_n \end{aligned}$$

1. Consider the sequence $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}$.

a) Let $\{a_n\}$ and $\{b_n\}$ be defined as above. Find the first few terms of both sequences.

b) Find $\limsup_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ and $\liminf_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$.

c) Find a subsequence that converges to something other than $\limsup_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ or $\liminf_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$.

2. Consider the sequence $\left\{\frac{\cos(n\pi/2)}{n}\right\}$.

a) Let $\{a_n\}$ and $\{b_n\}$ be defined as above. Find the first few terms of both sequences.

b) Find $\limsup_{n\to\infty} \frac{\cos(n\pi/2)}{n}$ and $\liminf_{n\to\infty} \frac{\cos(n\pi/2)}{n}$.

c) Do any subsequences converge to some other value?

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Definition. The sequence $\{x_n\}$ **converges** to $x \in \mathbb{R}$ if $\forall \varepsilon > 0\ \exists M \in \mathbb{N}\ \forall n \geq M,\ |x_n - x| < \varepsilon$.

Definition. A sequence $\{x_n\}$ is a **Cauchy sequence** if $\boxed{\forall \varepsilon > 0\ \exists M \in \mathbb{N}\ \forall n, k \geq m,\ |x_n - x_k| < \varepsilon}$

3. Show that $\left\{ \dfrac{n^2 - 1}{n^2} \right\}$ is a Cauchy sequence.