CONTINUITY

Definition. Let $S \subseteq \mathbb{R}$ and let $c \in S$ and $f : S \to \mathbb{R}$. The function f is **continuous at** c if

$$\boxed{\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in S\ \ (|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon)}$$

If f is continuous at all points of S, then we say that f is **continuous**.

Proposition (Basic properties of continuous functions)**.** *Let $S \subseteq \mathbb{R}$ and let $c \in S$ and $f : S \to \mathbb{R}$.*

i) If c is not a cluster point of S, then f is continuous at c.

ii) If c is a cluster point of S, then f is continuous at c if and only if the limit of $f(x)$ as $x \to c$ exists and $\lim_{x\to c} f(x) = f(c)$.

iii) The function f is continuous at c if and only if for every sequence $\{x_n\}$ having $x_n \in S$ for all n and $\lim_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

1. Determine if the following functions are continuous (pay attention to their domains). No proofs are required, though you should have some sense for how you might write a proof. You may use (without proof) the fact that the sine function is continuous.

a) $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sin\left(\frac{1}{x}\right)$

b) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

c) $f : \{0\} \cup [0.5, \infty) \to \mathbb{R}$ defined by $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

d) $f : \{0\} \cup \{\frac{2}{\pi n} : n \in \mathbb{N}\} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

e) $f : \{0\} \cup \{\frac{1}{\pi n} : n \in \mathbb{N}\} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

Date: Match 16, 2022.

2. Prove part i of the proposition.

3. Prove part ii of the proposition.

Challenge. Prove part iii of the proposition.