THE MVT

Proposition. *Let* $f : [a, b] \to \mathbb{R}$ *be differentiable at* $c \in (a, b)$*. If* f *has a relative minimum or a relative maximum at* c*, then* $f'(c) = 0$*.*

Theorem (Rolle's theorem)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous with* $f(a) = f(b)$*. If* f *is differentiable on* (a, b)*, then there is* $c \in (a, b)$ *such that* $f'(c) = 0$*.*

1. Prove Rolle's theorem.

2. Find an example of a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) = f(1)$ but $f'(c) \neq 0$ for every $c \in (0, 1)$ at which f is differentiable. Try to minimize the number of points at which f isn't differentiable.

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Theorem (Mean value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous and also differentiable on (a, b). Then there is $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

3. Your goal is to prove the MVT.

a) Find an equation $y = L(x)$ for the line through the points $(a, f(a))$ and $(b, f(b))$.
b) Define a new function $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - L(x)$.
c) Verify that g is continuous on $[a, b]$ and differentiable on (a, b).
d) Use Rolle's theorem and finish the proof.

Corollary. *Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable.*

i) f is constant if and only if $f'(x) = 0$ for all $x \in I$.
ii) f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.
iii) f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

4. Prove the corollary.

(you'll certainly need another sheet of paper)