THE FUNDAMENTAL THEOREM OF CALCULUS

Theorem (Fundamental Theorem of Calculus I)**.** *Let $F : [a, b] \to \mathbb{R}$ be continuous. If F is differentiable on (a, b), and $f \in \mathcal{R}[a, b]$ such that $f(x) = F'(x)$ for all $x \in (a, b)$, then $\int_a^b f = F(b) - F(a)$.*

1. Our goal is to prove the theorem. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$.

a) Apply the MVT to each interval $[x_{i-1}, x_i]$ to find $c_i \in (x_{i-1}, x_i)$.
b) Use the MVT to convert $\sum_{i=1}^n f(c_i)\Delta x_i$ to a sum involving F only (no derivatives). Cancel and simplify where possible.
c) How is $\sum_{i=1}^n f(c_i)\Delta x_i$ related to $L(P, f)$ and $U(P, f)$?
d) Take the supremum of $L(P, f)$ and infimum of $U(P, f)$ and use the fact that f is integrable to finish the proof.

Date: April 25, 2022.

Theorem (Fundamental Theorem of Calculus II)**.** *Let $f \in \mathcal{R}[a,b]$. Define $F(x) = \int_a^x f$. Then F is continuous on $[a,b]$ and if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.*

2. Our goal again is to prove the theorem.

a) First prove that F is Lipschitz continuous using the fact that f must be bounded.

b) Now suppose that f is continuous at c. Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$ (definition of continuity). Rewrite the last inequality with $f(x)$ in the middle.

c) Now use the bounds you found for f to find bounds for $\int_c^x f$ when $x > c$. (Careful with variables here: this x is different from the one before. Also watch out for $<$ becoming $\leq$.)

d) Rearrange your inequality to arrive at $-\varepsilon \leq$ ______ $\leq \varepsilon$.

e) Now use additivity of integrals ($\int_c^x f = \int_a^x f - \int_a^c f$) and the definition of F to finish proving that $\left| \dfrac{F(x) - F(c)}{x - c} - f(c) \right| \leq \varepsilon$.

f) Explain why the result follows even though we got $\leq$ instead of $<$.